# Lecture 4: Piggyback dualities

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## Outline



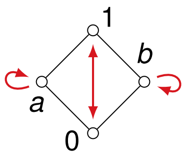


## A natural duality for De Morgan algebras

### Exercise

Use the Lattice-based Duality Theorem to find a natural duality for the class $\mathcal{A} = \mathbb{ISP}(\underline{\mathbf{M}})$ of De Morgan Algebras.

- De Morgan algebras. $\underline{\mathbf{M}} = \langle \{0, a, b, 1\}; \vee, \wedge, g, 0, 1\rangle$, where $\langle \{0, a, b, 1\}; \vee, \wedge, 0, 1\rangle$ is isomorphic to $\underline{\mathbf{D}}^2$ and $g$ is as shown below.




- In order to apply the Lattice-based Duality Theorem we need to find the lattice of subuniverses of $\underline{\mathbf{M}}^2$.
- Unfortunately, there are 55 subuniverses of $\underline{\mathbf{M}}^2$.
- There is a better way, and that is the topic of this lecture.



## Alter egos

We shall move now to the setting where structures are allowed on both sides. To simplify things, we restrict to total structures.

### An alter ego of a total structure

Let $\underline{\mathbf{M}} = \langle M; G_1, R_1\rangle$ be a total structure (possibly infinite). Then $\underset{\sim}{\mathbf{M}} = \langle M; G_2, R_2, \mathcal{T}\rangle$ is an alter ego of $\underline{\mathbf{M}}$ if it is compatible with $\underline{\mathbf{M}}$, that is,

- $G_2$ is a set of operations on $M$, each of which is a homomorphism with respect to $\underline{\mathbf{M}}$,
- $R_2$ is a set of relations on $M$, each of which is a subuniverse of the appropriate power of $\underline{\mathbf{M}}$, and
- $\mathcal{T}$ is a compact Hausdorff topology on $M$ with respect to which the operations $g \in G_1$ are continuous and the relations $r \in R_1$ are closed, i.e., $\underline{\mathbf{M}}^{\mathcal{T}} := \langle M; G_1, R_1, \mathcal{T}\rangle$ is a topological structure.

## The idea behind piggybacking

Assume that $\mathbf{\underline{M}} = \langle M; G_1, R_1 \rangle$ has a reduct $\mathbf{\underline{M}}^\flat$ in the class $\mathcal{D}$ of bounded distributive lattices, that is, there exist operations $\vee$, $\wedge$, 0 and 1 in $G_1$ such that $\mathbf{\underline{M}}^\flat := \langle M; \vee, \wedge, 0, 1 \rangle$ belongs to $\mathcal{D}$. As usual, define $\mathcal{A} = \mathsf{ISP}(\mathbf{\underline{M}})$. It follows that

- we have a forgetful functor $^\flat \colon \mathcal{A} \to \mathcal{D}$,
- for each $\omega \in \mathcal{D}(\mathbf{\underline{M}}^\flat, \mathbf{\underline{D}})$ and $\mathbf{A} \in \mathcal{A}$, we may define a map

  $$\Phi_\omega^{\mathbf{A}} \colon \mathcal{A}(\mathbf{A}, \mathbf{\underline{M}}) \to \mathcal{D}(\mathbf{A}^\flat, \mathbf{\underline{D}})$$

  by $\Phi_\omega^{\mathbf{A}}(x) := \omega \circ x$, for all $x \in \mathcal{A}(\mathbf{A}, \mathbf{\underline{M}})$.

The idea is to use some or all of the maps $\Phi_\omega^{\mathbf{A}}$ and the fact that $\underset{\sim}{\mathbf{D}} = \langle \{0,1\}; \leqslant, \mathcal{T} \rangle$ yields a duality on $\mathcal{D}$ to define an alter ego $\underset{\sim}{\mathbf{M}}$ that yields a duality on $\mathcal{A}$.



## Some notation

Let $\underset{\sim}{\mathbf{M}} = \langle M; G_2, R_2, \mathcal{T} \rangle$ be an alter ego go $\mathbf{\underline{M}}$.

- $\mathrm{Clo}_1(\underset{\sim}{\mathbf{M}})$ denotes the set of unary term functions of $\underset{\sim}{\mathbf{M}}$.

  Since $\underset{\sim}{\mathbf{M}}$ is compatible with $\mathbf{\underline{M}}$, we have $\mathrm{Clo}_1(\underset{\sim}{\mathbf{M}}) \subseteq \mathrm{End}(\mathbf{\underline{M}})$.
- Let $\Omega \subseteq \mathcal{D}(\mathbf{\underline{M}}^\flat, \mathbf{\underline{D}})$. We define

  $$\Omega \circ \mathrm{Clo}_1(\underset{\sim}{\mathbf{M}}) := \{\, \omega \circ u \mid \omega \in \Omega \;\;\&\;\; u \in \mathrm{Clo}_1(\underset{\sim}{\mathbf{M}}) \,\} \subseteq \{0,1\}^M.$$
- If $\Omega = \{\omega\}$, then we write simply $\omega \circ \mathrm{Clo}_1(\underset{\sim}{\mathbf{M}})$.
- We say that $\Omega \circ \mathrm{Clo}_1(\underset{\sim}{\mathbf{M}})$ separates the points of $M$ if, for all $a, b \in M$ with $a \neq b$, there exits $\omega \in \Omega$ and $u \in \mathrm{Clo}_1(\underset{\sim}{\mathbf{M}})$ with $\omega(u(a)) \neq \omega(u(b))$.



## Some further notation

Recall that $\Omega \subseteq \mathcal{D}(\mathbf{\underline{M}}^\flat, \mathbf{\underline{D}})$.

- For $\omega_1, \omega_2 \in \Omega$, define

  $$(\omega_1, \omega_2)^{-1}(\leqslant) := \{\, (a,b) \in M^2 \mid \omega_1(a) \leqslant \omega_2(b) \,\}.$$
- Define

  $$\max\nolimits_{\mathbf{\underline{M}}} \Omega^{-1}(\leqslant) := \{\, s \subseteq M^2 \mid \mathbf{s} \leqslant \mathbf{\underline{M}}^2 \text{ with } s \text{ maximal in } (\omega_1, \omega_2)^{-1}(\leqslant) \text{ for some } \omega_1, \omega_2 \in \Omega \,\}$$



## $\mathcal{D}$-based Piggyback Duality Theorem

**Theorem ($\mathcal{D}$-based Piggyback Duality Theorem)**

*Let $\mathbf{\underline{M}}$ be a $\mathcal{D}$-based total structure with reduct $\mathbf{\underline{M}}^\flat$ in $\mathcal{D}$. Then an alter ego $\underset{\sim}{\mathbf{M}}$ of $\mathbf{\underline{M}}$ dualises $\mathbf{\underline{M}}$ provided that there is a finite subset $\Omega$ of $\mathcal{D}(\mathbf{\underline{M}}^\flat, \mathbf{\underline{D}})$ such that*

(0) *each $\omega \in \Omega$ is continuous with respect to the topologies on $\underset{\sim}{\mathbf{M}}$ and $\underset{\sim}{\mathbf{D}}$,*

(1) *$\Omega \circ \mathrm{Clo}_1(\underset{\sim}{\mathbf{M}})$ separates the points of $M$, and*

(2) *$\underset{\sim}{\mathbf{M}}$ entails every relation in $\max_{\mathbf{\underline{M}}} \Omega^{-1}(\leqslant)$.*

**Remark 1**

When $\mathbf{\underline{M}}$ is finite, this gives us a recipe for $\underset{\sim}{\mathbf{M}} = \langle M; G_2, R_2, \mathcal{T} \rangle$:

- choose $G_2 = \mathrm{End}(\mathbf{\underline{M}})$,
- choose $\Omega \subseteq \mathcal{D}(\mathbf{\underline{M}}^\flat, \mathbf{\underline{D}})$ so that $\Omega \circ \mathrm{End}(\mathbf{\underline{M}})$ separates the points of $M$. (The choice $\Omega = \mathcal{D}(\mathbf{\underline{M}}^\flat, \mathbf{\underline{D}})$ always works.)
- choose $R_2 = \max_{\mathbf{\underline{M}}} \Omega^{-1}(\leqslant)$.

## Remarks continued

Assume that $\mathbf{M}$ is finite.

### Remark 1

The theorem gives us a recipe for $\underset{\sim}{\mathbf{M}} = \langle M; G_2, R_2, \mathcal{T}\rangle$:

- choose $G_2 = \mathrm{End}(\mathbf{M})$,
- choose $\Omega \subseteq \mathcal{D}(\mathbf{M}^\flat, \mathbf{D})$ so that $\Omega \circ \mathrm{End}(\mathbf{M})$ separates the points of $M$. (The choice $\Omega = \mathcal{D}(\mathbf{M}^\flat, \mathbf{D})$ always works.)
- choose $R_2 = \max_{\mathbf{M}} \Omega^{-1}(\leqslant)$.

### Remark 2

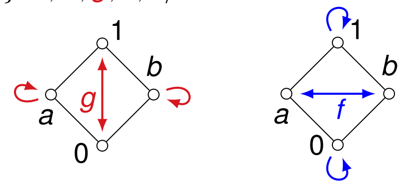
- Choose $\Omega = \mathcal{D}(\mathbf{M}^\flat, \mathbf{D})$. Then $\underset{\sim}{\mathbf{M}} := \langle M; \max_{\mathbf{M}} \Omega^{-1}(\leqslant), \mathcal{T}\rangle$ yields a duality on $\mathcal{A} = \mathrm{ISP}(\mathbf{M})$.
- To minimise the size of $R_2 = \max_{\mathbf{M}} \Omega^{-1}(\leqslant)$ we should use $G_2 = \mathrm{End}(\mathbf{M})$ and choose $\Omega \subseteq \mathcal{D}(\mathbf{M}^\flat, \mathbf{D})$ as small as possible.



## De Morgan algebras again

### A natural duality for De Morgan algebras

The class of De Morgan Algebras equals $\mathcal{A} = \mathrm{ISP}(\mathbf{M})$, where $\mathbf{M} = \langle \{0, a, b, 1\}; \vee, \wedge, g, 0, 1\rangle$:




Choose $f$ to be the automorphism of $\mathbf{M}$ shown on the right and let $\omega\colon \mathbf{M}^\flat \to \mathbf{D}$ be the map with kernel $\{0, a \mid b, 1\}$.

It is clear that $\omega \circ \{\mathrm{id}_M, f\}$ separates the points of $M$.

It remains to calculate $\max_{\mathbf{M}}\{\omega\}^{-1}(\leqslant)$, i.e., the maximal De Morgan subuniverses of

$$(\omega, \omega)^{-1}(\leqslant) = \{\, (a, b) \in M^2 \mid \omega(a) \leqslant \omega(b) \,\}.$$



## De Morgan algebras again

### A very useful result

### Lemma

*Let $\mathbf{A} = \langle A; \vee, \wedge, g, 0, 1\rangle$, where $g$ is either*

- *an endomorphism of $\mathbf{A}^\flat$, or*
- *a dual-endomorphism of $\mathbf{A}^\flat$.*

*Let $\mathbf{L}$ be a $\{0, 1\}$-sublattice of $\mathbf{A}^\flat$, then there is a largest subuniverse $L^\circ$ of $\mathbf{A}$ satisfying $L^\circ \subseteq L$. Indeed,*

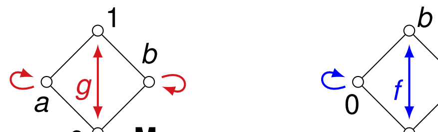
$$L^\circ := L \backslash \{\, a \in L \mid (\exists k \geqslant 1)\ g^k(a) \notin L \,\}.$$

Since $(\omega, \omega)^{-1}(\leqslant) := \{\, (a, b) \in M^2 \mid \omega(a) \leqslant \omega(b) \,\}$ is a $\{0, 1\}$-sublattice of $\mathbf{M}^\flat$, it follows from the lemma that there is a unique De Morgan subuniverse of $\mathbf{M}^2$ that is maximal in $(\omega, \omega)^{-1}(\leqslant)$, namely $(\omega, \omega)^{-1}(\leqslant)^\circ$.



## De Morgan algebras again

### Calculating $\max_{\mathbf{M}}\{\omega\}^{-1}(\leqslant)$




$\mathbf{M}$ $\quad$ $\underset{\sim}{\mathbf{M}} := \langle M; f, \preccurlyeq, \mathcal{T}\rangle$

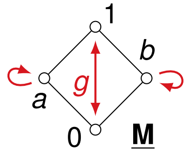
$\omega\colon \mathbf{M}^\flat \to \mathbf{D}$ is the map with kernel $\{0, a \mid b, 1\}$. Hence

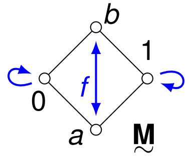
$$\begin{aligned}(\omega, \omega)^{-1}(\leqslant) &= \{0, a\} \times \{0, a, b, 1\} \cup \{b, 1\} \times \{b, 1\}\\ &= M^2 \backslash \{b0, ba, 10, 1a\}\\ &= \{00, 0a, 0b, 01, a0, aa, ab, a1, bb, b1, 1b, 11\}\end{aligned}$$

Hence

$$(\omega, \omega)^{-1}(\leqslant)^\circ = \{00, 0b, a0, aa, ab, a1, bb, 1b, 11\} = \preccurlyeq$$

## De Morgan algebras again




**Theorem (Cornish and Fowler)**

*$\underset{\sim}{\mathbf{M}} := \langle M; f, \preccurlyeq, \mathcal{T}\rangle$ yields a duality on the class $\mathcal{A} = \mathrm{ISP}(\underline{\mathbf{M}})$ of De Morgan algebras. (The duality is strong.)*

- The dual category $\mathcal{X} = \mathrm{IS_cP^+}(\underset{\sim}{\mathbf{M}})$ is the category of Priestley spaces with a continuous order-reversing map $f$ satisfying $f^2 = \mathrm{id}$.
- Surprisingly, this is precisely the same as the restricted Priestley duality for De Morgan algebras. (More on this later.)



## Natural dualities for Gödel algebras of degree $n$

Let $\underline{\mathbf{C}}_n = \langle C_n; \vee, \wedge, \to, 0, 1\rangle$ be the $n$-element chain regarded as a Heyting algebra. Thus,

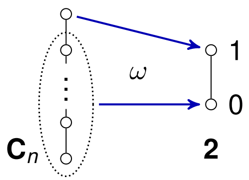
$$a \to b = \begin{cases} 1 & \text{if } a \leqslant b, \\ 0 & \text{if } a > b. \end{cases}$$

The class $\mathcal{G}_n := \mathrm{ISP}(\underline{\mathbf{C}}_n)$ is the class of Gödel algebras of degree $n$.

- Define $\omega\colon \underline{\mathbf{C}}_n^\flat \to \underline{\mathbf{D}}$ by $\omega = \chi_{\{1\}}$.
- For all $a < b$ in $\underline{\mathbf{C}}_n$, there exists $u \in \mathrm{End}(\underline{\mathbf{C}}_n)$ with $u(a) < u(b) = 1$.
- Hence $\omega \circ \mathrm{End}(\underline{\mathbf{C}}_n)$ separates the points of $C_n$.
- Thus $\underset{\sim}{\mathbf{C}}_n := \langle C_n; \mathrm{End}(\underline{\mathbf{C}}_n), \max_{\underline{\mathbf{C}}_n}\{\omega\}^{-1}(\leqslant), \mathcal{T}\rangle$ yields a duality on $\mathcal{G}_n$.



## Natural dualities for Gödel algebras of degree $n$




Now let $\mathbf{r} \leqslant \underset{\sim}{\mathbf{C}}_n^2$ with $r \subseteq (\omega, \omega)^{-1}(\leqslant)$. Claim: $r$ is the graph of a partial endomorphism of $\underline{\mathbf{C}}_n$. Let $(a, b), (a, c) \in r$. Then

$$\begin{aligned} (a,b),(a,c) \in r &\implies (1, b \to c) = (a,b) \to (a,c) \in r \\ &\implies 1 = \omega(1) \leqslant \omega(b \to c) \\ &\implies \omega(b \to c) = 1 \\ &\implies b \to c = 1 \\ &\implies b \leqslant c \quad \text{and } c \leqslant b, \text{ by symmetry} \\ &\implies b = c \end{aligned}$$

Hence $r$ is the graph of a partial endomorphism of $\underline{\mathbf{C}}_n$. $\square$



## Natural dualities for Gödel algebras

We now know that $\underset{\sim}{\mathbf{C}}_n := \langle C_n; \mathrm{End}(\underline{\mathbf{C}}_n), H, \mathcal{T}\rangle$ yields a duality on $\mathcal{G}_n$, where $H$ is the set of proper partial endomorphisms of $\underline{\mathbf{C}}_n$.

In fact, $H$ can be removed without destroying the duality, i.e.,

$\mathrm{End}(\underline{\mathbf{C}}_n)$ entails every $h \in H$.

The shortest proof of this uses three straightforward general results.

## Three easy exercises for you

Let $\mathbf{M}$ be a finite algebra.

Corollary 8.1.4

An alter ego $\underset{\sim}{\mathbf{M}}$ of $\mathbf{M}$ entails an algebraic relation $s$ provided $\underset{\sim}{\mathbf{M}}$ yields a duality on some isomorphic copy of the algebra $\mathbf{s}$.

Exercise 2.3

Assume that $\underset{\sim}{\mathbf{M}}$ is an alter ego of $\mathbf{M}$ that yields a duality on $\mathbf{A} \in \mathsf{ISP}(\mathbf{M})$, then $\underset{\sim}{\mathbf{M}}$ yields a duality on every retract of $\mathbf{A}$.

Exercise 2.4

$\underset{\sim}{\mathbf{M}} = \langle M; \mathrm{End}(\mathbf{M}), \mathcal{T}\rangle$ yields a duality on the algebra $\mathbf{M}$.

Time permitting, I will explain how it follows easily from these results that $\underset{\sim}{\mathbf{C}}_n = \langle C_n; \mathrm{End}(\mathbf{M}), \mathcal{T}\rangle$ entails every partial endomorphism of $\mathbf{C}_n$.



## Natural dualities for Gödel algebras of degree $n$

Theorem (Davey, a long time ago)

*$\underset{\sim}{\mathbf{C}}_n := \langle C_n; \mathrm{End}(\mathbf{C}_n), \mathcal{T}\rangle$ yields a duality on the class $\mathcal{G}_n$ of Gödel algebras of degree n. Hence, $\mathbf{C}_n$ is endo-dualisable.*

- For $n \geqslant 4$, this duality is not strong.
- It can be made strong by adding back the partial endomorphisms.
- It provides a good example of when a non-full duality might be easier to use that a full or strong duality.



## $\mathcal{D}$-based Piggyback Strong Duality Theorem

Notation

- Denote the functors that give the Priestley duality between the category $\mathcal{D} = \mathsf{ISP}(\mathbf{D})$ of bounded distributive lattices and the category $\mathcal{P} = \mathsf{IS_cP^+}(\underset{\sim}{\mathbf{D}})$ of Priestley spaces by

$$H\colon \mathcal{D} \to \mathcal{P} \text{ and } K\colon \mathcal{P} \to \mathcal{D}.$$

- Thus, on objects, we have

$$H(\mathbf{A}) = \mathcal{D}(\mathbf{A}, \mathbf{D}) \text{ and } K(\mathbf{X}) = \mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{D}}),$$

  for all $\mathbf{A} \in \mathcal{D}$ and $\mathbf{X} \in \mathcal{P}$.



## $\mathcal{D}$-based Piggyback Strong Duality Theorem

Theorem (Davey, Haviar, Priestley 2015)

*Let $\mathbf{M}$ be a total structure with reduct $\mathbf{M}^\flat$ in $\mathcal{D}$, let $\underset{\sim}{\mathbf{M}}$ be an alter ego of $\mathbf{M}$ and define $\mathcal{A} := \mathsf{ISP}(\mathbf{M})$ and $\mathcal{X} := \mathsf{IS_cP^+}(\underset{\sim}{\mathbf{M}})$. Assume that the structure on $\underset{\sim}{\mathbf{M}}$ includes an order relation $\leqslant$ such that $\underset{\sim}{\mathbf{M}}^\flat := \langle M; \leqslant, \mathcal{T}\rangle$ is a Priestley space, and there exists $\omega \in \mathcal{D}(\mathbf{M}^\flat, \mathbf{D}) \cap \mathcal{P}(\underset{\sim}{\mathbf{M}}^\flat, \underset{\sim}{\mathbf{D}})$ such that*

(1) *$\omega \circ \mathrm{Clo}_1(\mathbf{M})$ separates the points of $M$,*

(2) *$\underset{\sim}{\mathbf{M}}$ entails each (binary) relation in $\max_{\mathbf{M}}\{\omega\}^{-1}(\leqslant)$, and*

(3) *if $x \nleqslant y$ in $\underset{\sim}{\mathbf{M}}^\flat$, then there exists $t \in \mathrm{Clo}_1(\mathbf{M})$ such that $\omega(t(x)) = 1$ and $\omega(t(y)) = 0$.*

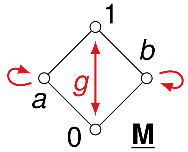
*Then*

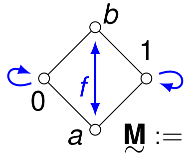
(a) *$\underset{\sim}{\mathbf{M}}$ fully dualises $\mathbf{M}$,*

(b) *$\mathbf{M}$ is injective in $\mathcal{A}$ and $\underset{\sim}{\mathbf{M}}$ is injective in $\mathcal{X}$, and*

(c) *$D(\mathbf{A})^\flat \cong H(\mathbf{A}^\flat)$ and $E(\mathbf{X})^\flat \cong K(\mathbf{X}^\flat)$, for all $\mathbf{A} \in \mathcal{A}$, $\mathbf{X} \in \mathcal{X}$.*

## De Morgan algebras yet again




$\underset{\sim}{\mathbf{M}} := \langle M; f, \preccurlyeq, \mathcal{T} \rangle$

$\omega \colon \mathbf{M}^\flat \to \mathbf{D}$ is the map with kernel $\{0, a \mid b, 1\}$.

We already know that $\underset{\sim}{\mathbf{M}}$ satisfies conditions (1) and (2) of the $\mathcal{D}$-based Piggyback Strong Duality Theorem.

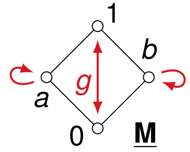
(3) if $x \not\preccurlyeq y$ in $\underset{\sim}{\mathbf{M}}^\flat$, then there exists $t \in \mathrm{Clo}_1(\mathbf{M})$ such that $\omega(t(x)) = 1$ and $\omega(t(y)) = 0$.

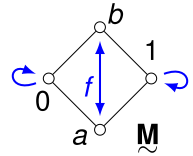
There are 7 pairs to check. Some examples:

$$\begin{array}{ll} b \not\preccurlyeq a: & \omega(b) = 1 \ \& \ \omega(a) = 0 \\ b \not\preccurlyeq 1: & \omega(g(b)) = 1 \ \& \ \omega(g(1)) = 0 \\ 0 \not\preccurlyeq 1: & \omega(g(0)) = 1 \ \& \ \omega(g(1)) = 0 \end{array}$$



## De Morgan algebras yet again




$\underset{\sim}{\mathbf{M}}$

### Theorem (Cornish and Fowler)

- *$\underset{\sim}{\mathbf{M}} := \langle M; f, \preccurlyeq, \mathcal{T} \rangle$ yields a strong duality between the category $\mathcal{A} = \mathrm{ISP}(\mathbf{M})$ of De Morgan algebras and the category $\mathfrak{X} = \mathrm{IS_cP^+}(\underset{\sim}{\mathbf{M}})$.*
- *The underlying ordered space of the natural dual is the Priestley dual:*

  $$D(\mathbf{A})^\flat \cong H(\mathbf{A}^\flat) \ \text{ and } \ E(\mathbf{X})^\flat \cong K(\mathbf{X}^\flat),$$

  *for all $\mathbf{A} \in \mathcal{A}$ and all $\mathbf{X} \in \mathfrak{X}$.*



## Piggyback duality for Ockham algebras

### Ockham algebras

$\mathbf{A} = \langle A; \vee, \wedge, g, 0, 1 \rangle$ is an Ockham algebra if $\mathbf{A}^\flat \in \mathcal{D}$ and $g$ is a lattice-dual endomorphism of $\mathbf{A}^\flat$. We denote the equational class of Ockham algebras by $\mathcal{O}$.

- Let $\gamma \colon \mathbb{N}_0 \to \mathbb{N}_0$ be the successor function: $\gamma(n) := n + 1$ and let $c$ denote Boolean complementation on $\{0, 1\}$.
- Define $\mathbf{M}_1 := \langle \{0, 1\}^{\mathbb{N}_0} \mid \vee, \wedge, g, \underline{0}, \underline{1} \rangle$, where
  - $\vee$ and $\wedge$ are defined pointwise, $\underline{0}$ and $\underline{1}$ are the constant maps onto 0 and 1, respectively, and,
  - for all $a \in \{0, 1\}^{\mathbb{N}_0}$ we have $g(a) := c \circ a \circ \gamma$. Thus, $g$ is given by shift left and then negate; for example,

    $$g(0110010\ldots) = (001101\ldots).$$

Then $\mathbf{M}_1$ is an Ockham algebra. Moreover, $\mathcal{O} = \mathrm{ISP}(\mathbf{M}_1)$.



## Piggyback duality for Ockham algebras

### An alter ego $\underset{\sim}{\mathbf{M}}_2^{\mathcal{T}} = \langle \{0, 1\}^{\mathbb{N}_0}; u, \preccurlyeq, \mathcal{T} \rangle$

- $u \colon \{0, 1\}^{\mathbb{N}_0} \to \{0, 1\}^{\mathbb{N}_0}$ is the left shift operator, given by $u(a) := a \circ \gamma$. Thus, for example,

  $$u(0110010\ldots) = (110010\ldots).$$

  Then $u \in \mathrm{End}(\mathbf{M}_1)$.
- $\preccurlyeq$ is the alternating order on $\{0, 1\}^{\mathbb{N}_0}$, that is, for all $a, b \in \{0, 1\}^{\mathbb{N}_0}$,

  $$a \preccurlyeq b \iff a(0) \leqslant b(0) \ \& \ a(1) \geqslant b(1) \ \& \ a(2) \leqslant b(2) \ \& \ \cdots.$$

- $\mathcal{T}$ is the product topology on $\{0, 1\}^{\mathbb{N}_0}$ coming from the discrete topology on $\{0, 1\}$.

## Piggyback duality for Ockham algebras

Let $\omega := \pi_0 \colon \{0,1\}^{\mathbb{N}_0} \to \{0,1\}$.

Then, $\pi_0 \in \mathcal{D}(\underline{\mathbf{M}}_1^\flat, \underline{\mathbf{D}}) \cap \mathcal{P}((\underline{\mathbf{M}}_2^\flat)^{\mathcal{T}}, \underset{\sim}{\mathbf{D}})$.

We now check Conditions (1)–(3) of the $\mathcal{D}$-based Piggyback Strong Duality Theorem.

(1) The set $\pi_0 \circ \mathrm{Clo}_1(\underline{\mathbf{M}}_2)$ separates the points of $M = \{0,1\}^{\mathbb{N}_0}$: indeed, let $a, b \in \{0,1\}^{\mathbb{N}_0}$ with $a \neq b$, then

$$\begin{aligned} & a \neq b \\ \Longrightarrow\ & (\exists n \in \mathbb{N}_0)\, a(n) \neq b(n) \\ \Longrightarrow\ & (\exists n \in \mathbb{N}_0)\, u^n(a)(0) = (a \circ \gamma^n)(0) \neq (b \circ \gamma^n)(0) = u^n(b)(0) \\ \Longrightarrow\ & (\exists n \in \mathbb{N}_0)\, (\pi_0 \circ u^n)(a) \neq (\pi_0 \circ u^n)(b). \end{aligned}$$

As $u$ is in $\mathrm{Clo}_1(\underline{\mathbf{M}}_2)$, so is $u^n$. Hence $\pi_0 \circ \mathrm{Clo}_1(\underline{\mathbf{M}}_2)$ separates the points of $M$, that is, Condition (1) holds.



## Piggyback duality for Ockham algebras

(2) We must find the binary relations $r$ on $M$ which form substructures of $\underline{\mathbf{M}}_1^2$ that are maximal in $(\pi_0, \pi_0)^{-1}(\leqslant)$. We have

$$(\pi_0, \pi_0)^{-1}(\leqslant) = \{\, (a, b) \in (\{0,1\}^{\mathbb{N}_0})^2 \mid a(0) \leqslant b(0) \,\}.$$

Let $\mathbf{r}$ be a subalgebra of $\underline{\mathbf{M}}_1^2$ with $r \subseteq (\pi_0, \pi_0)^{-1}(\leqslant)$. Then

$$\begin{aligned} (a, b) \in r \ &\Longrightarrow\ (\forall n \in \mathbb{N}_0)\, (g^n(a), g^n(b)) \in r \\ &\Longrightarrow\ (\forall n \in \mathbb{N}_0)\, g^n(a)(0) \leqslant g^n(b)(0) \\ &\Longrightarrow\ a(0) \leqslant b(0) \ \ \&\ \ a(1) \geqslant b(1) \ \ \&\ \ a(2) \leqslant b(2) \ \ \&\ \cdots \\ &\Longleftrightarrow\ a \preccurlyeq b. \end{aligned}$$

Thus $r \subseteq \preccurlyeq$. Since $\preccurlyeq$ forms a subalgebra of $\underline{\mathbf{M}}_1^2$ and $\preccurlyeq \subseteq (\pi_0, \pi_0)^{-1}(\leqslant)$, it follows that $(\pi_0, \pi_0)^{-1}(\leqslant)^{\circ} = \preccurlyeq$.



## Piggyback duality for Ockham algebras

(3) We must prove that $\pi_0 \circ \mathrm{Clo}_1(\underline{\mathbf{M}}_1)$ separates the relation $\preccurlyeq$, that is, if $a \not\preccurlyeq b$ in $\underline{\mathbf{M}}_2^\flat$, then there exists $t \in \mathrm{Clo}_1(\underline{\mathbf{M}}_1)$ such that $\pi_0(t(a)) = 1$ and $\pi_0(t(b)) = 0$. We have

$$\begin{aligned} & a \not\preccurlyeq b \text{ in } \underline{\mathbf{M}}_2^\flat \\ \Longleftrightarrow\ & (\exists n \in \mathbb{N}_0) \begin{cases} a(n) = 1 \ \ \&\ \ b(n) = 0, & \text{if } n \text{ is even} \\ a(n) = 0 \ \ \&\ \ b(n) = 1, & \text{if } n \text{ is odd} \end{cases} \\ \Longrightarrow\ & (\exists n \in \mathbb{N}_0)\, g^n(a)(0) = 1 \ \ \&\ \ g^n(b)(0) = 0 \\ \Longrightarrow\ & (\exists n \in \mathbb{N}_0)\, \pi_0(g^n(a)) = 1 \ \ \&\ \ \pi_0(g^n(b)) = 0 \end{aligned}$$

as required, with $t(v) := g^n(v)$.



## Piggyback duality for Ockham algebras

**Theorem (Goldberg 1981/1983)**

*Let* $\underline{\mathbf{M}}_1 := \langle \{0,1\}^{\mathbb{N}_0} \mid \vee, \wedge, g, \underline{0}, \underline{1} \rangle$ *and* $\underline{\mathbf{M}}_2^{\mathcal{T}} = \langle \{0,1\}^{\mathbb{N}_0}; u, \preccurlyeq, \mathcal{T} \rangle$.

- $\underline{\mathbf{M}}_2^{\mathcal{T}}$ *yields a strong duality between the category* $\mathcal{O} = \mathrm{ISP}(\underline{\mathbf{M}}_1)$ *of Ockham algebras and the category* $\mathcal{Y} = \mathrm{IS_cP^+}(\underset{\sim}{\mathbf{M}}_2^{\mathcal{T}})$ *of Ockham spaces.*
- *The underlying ordered space of the natural dual is the Priestley dual:*

$$D(\mathbf{A})^\flat \cong H(\mathbf{A}^\flat) \ \text{ and } \ E(\mathbf{X})^\flat \cong K(\mathbf{X}^\flat),$$

*for all* $\mathbf{A} \in \mathcal{A}$ *and all* $\mathbf{X} \in \mathcal{X}$.

## Two-for-one piggyback duality for Ockham algebras

We get a second strong duality for free simply by swapping the topology from one structure to the other.

**Theorem (Davey, Haviar, Priestley 2015)**

*Let* $\underline{\mathbf{M}}_1 := \langle \{0,1\}^{\mathbb{N}_0} \mid \vee, \wedge, g, \underline{0}, \underline{1} \rangle$ *and* $\underline{\mathbf{M}}_2 = \langle \{0,1\}^{\mathbb{N}_0}; u, \preccurlyeq \rangle$.

*Then*

- $\underline{\mathbf{M}}_2^{\mathcal{T}} := \langle \{0,1\}^{\mathbb{N}_0}; u, \preccurlyeq, \mathcal{T} \rangle$ *strongly dualises* $\underline{\mathbf{M}}_1$, *and*
- $\underline{\mathbf{M}}_1^{\mathcal{T}} := \langle \{0,1\}^{\mathbb{N}_0}; \vee, \wedge, \neg, \underline{0}, \underline{1}, \mathcal{T} \rangle$ *strongly dualises* $\underline{\mathbf{M}}_2$.

Note that

- $\mathsf{ISP}(\underline{\mathbf{M}}_2)$ consists of ordered sets equipped with an order-reversing map, and
- $\mathsf{IS_cP^+}(\underline{\mathbf{M}}_1^{\mathcal{T}})$ consists of Boolean-topological Ockham algebras.



## Some homework exercises for you

**Some theory**

- Prove the claims in Corollary 8.1.4, Exercise 2.3 and Exercise 2.4.

**Some practice**

In each case, use the useful lemma and compare your answer to the duality we have already found.

- Use the two-element set $\Omega = \mathcal{D}(\underline{\mathbf{K}}^\flat, \underline{\mathbf{D}})$ to obtain a duality for Kleene algebras via the Piggyback Duality Theorem.
- Use the two-element set $\Omega = \mathcal{D}(\underline{\mathbf{M}}^\flat, \underline{\mathbf{D}})$ to obtain a duality for De Morgan algebras via the Piggyback Duality Theorem.