Lecture 4: Piggyback dualities

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TACL 2015 School
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Outline

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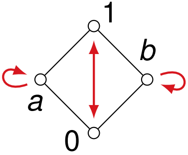
A natural duality for De Morgan algebras

Exercise

Use the Lattice-based Duality Theorem to find a natural duality for the class $\mathcal{A} = \mathrm{ISP}(\underline{\mathbf{M}})$ of De Morgan Algebras.

- De Morgan algebras. $\underline{\mathbf{M}} = \langle\{0, a, b, 1\}; \vee, \wedge, g, 0, 1\rangle$, where $\langle\{0, a, b, 1\}; \vee, \wedge, 0, 1\rangle$ is isomorphic to $\underline{\mathbf{D}}^2$ and g is as shown below.



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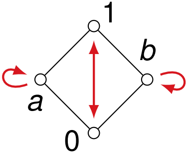

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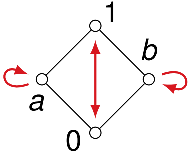

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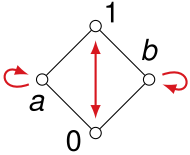

- In order to apply the Lattice-based Duality Theorem we need to find the lattice of subuniverses of $\underline{\mathbf{M}}^2$.
- Unfortunately, there are 55 subuniverses of $\underline{\mathbf{M}}^2$.
- There is a better way, and that is the topic of this lecture.

Outline

Alter egos

We shall move now to the setting where structures are allowed on both sides. To simplify things, we restrict to total structures.

An alter ego of a total structure

Let $\underline{\mathbf{M}} = \langle M; G_1, R_1 \rangle$ be a total structure (possibly infinite). Then $\underset{\sim}{\mathbf{M}} = \langle M; G_2, R_2, \mathcal{T} \rangle$ is an alter ego of $\underline{\mathbf{M}}$ if it is compatible with $\underline{\mathbf{M}}$, that is,

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- G_2 is a set of operations on M, each of which is a homomorphism with respect to $\underline{\mathbf{M}}$,
- R_2 is a set of relations on M, each of which is a subuniverse of the appropriate power of $\underline{\mathbf{M}}$, and
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- $\mathcal{T}$ is a compact Hausdorff topology on M with respect to which the operations $g \in G_1$ are continuous and the relations $r \in R_1$ are closed,
  i.e., $\underline{\mathbf{M}}^{\mathcal{T}} := \langle M; G_1, R_1, \mathcal{T} \rangle$ is a topological structure.

The idea behind piggybacking

Assume that $\underline{\mathbf{M}} = \langle M; G_1, R_1 \rangle$ has a reduct $\underline{\mathbf{M}}^\flat$ in the class $\mathcal{D}$ of bounded distributive lattices, that is, there exist operations $\vee$, $\wedge$, 0 and 1 in G_1 such that $\underline{\mathbf{M}}^\flat := \langle M; \vee, \wedge, 0, 1 \rangle$ belongs to $\mathcal{D}$. As usual, define $\mathcal{A} = \mathsf{ISP}(\underline{\mathbf{M}})$. It follows that

- we have a forgetful functor ${}^\flat \colon \mathcal{A} \to \mathcal{D}$,
- for each $\omega \in \mathcal{D}(\underline{\mathbf{M}}^\flat, \underline{\mathbf{D}})$ and $\mathbf{A} \in \mathcal{A}$, we may define a map

  $$\Phi^{\mathbf{A}}_\omega \colon \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}}) \to \mathcal{D}(\mathbf{A}^\flat, \underline{\mathbf{D}})$$

  by $\Phi^{\mathbf{A}}_\omega(x) := \omega \circ x$, for all $x \in \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$.

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$$\Phi_\omega^{\mathbf{A}}\colon \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}}) \to \mathcal{D}(\mathbf{A}^\flat, \underline{\mathbf{D}})$$

by $\Phi_\omega^{\mathbf{A}}(x) := \omega \circ x$, for all $x \in \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$.

The idea is to use some or all of the maps $\Phi_\omega^{\mathbf{A}}$ and the fact that $\underset{\sim}{\mathbf{D}} = \langle \{0,1\}; \leqslant, \mathcal{T} \rangle$ yields a duality on $\mathcal{D}$ to define an alter ego $\underset{\sim}{\mathbf{M}}$ that yields a duality on $\mathcal{A}$.

Some notation

Let $\underset{\sim}{\mathbf{M}} = \langle M; G_2, R_2, \mathcal{T} \rangle$ be an alter ego go $\underline{\mathbf{M}}$.

- $\mathrm{Clo}_1(\underset{\sim}{\mathbf{M}})$ denotes the set of unary term functions of $\underset{\sim}{\mathbf{M}}$.

  Since $\underset{\sim}{\mathbf{M}}$ is compatible with $\underline{\mathbf{M}}$, we have $\mathrm{Clo}_1(\underset{\sim}{\mathbf{M}}) \subseteq \mathrm{End}(\underline{\mathbf{M}})$.

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  $$\Omega \circ \mathrm{Clo}_1(\underset{\sim}{\mathbf{M}}) := \{\, \omega \circ u \mid \omega \in \Omega \;\;\&\;\; u \in \mathrm{Clo}_1(\underset{\sim}{\mathbf{M}}) \,\} \subseteq \{0, 1\}^M.$$

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- We say that $\Omega \circ \mathrm{Clo}_1(\underset{\sim}{\mathbf{M}})$ separates the points of M if, for all $a, b \in M$ with $a \neq b$, there exits $\omega \in \Omega$ and $u \in \mathrm{Clo}_1(\underset{\sim}{\mathbf{M}})$ with $\omega(u(a)) \neq \omega(u(b))$.

Some further notation

Recall that $\Omega \subseteq \mathfrak{D}(\underline{\mathbf{M}}^{\flat}, \underline{\mathbf{D}})$.

- For $\omega_1, \omega_2 \in \Omega$, define

$$(\omega_1, \omega_2)^{-1}(\leqslant) := \{ (a, b) \in M^2 \mid \omega_1(a) \leqslant \omega_2(b) \}.$$

Some further notation

Recall that $\Omega \subseteq \mathfrak{D}(\underline{\mathbf{M}}^\flat, \underline{\mathbf{D}})$.

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- Define

$$\max_{\underline{\mathbf{M}}} \Omega^{-1}(\leqslant) := \{\, s \subseteq M^2 \mid \mathbf{s} \leqslant \underline{\mathbf{M}}^2 \text{ with } s \text{ maximal in } (\omega_1, \omega_2)^{-1}(\leqslant) \text{ for some } \omega_1, \omega_2 \in \Omega \,\}$$

$\mathfrak{D}$-based Piggyback Duality Theorem

Theorem ($\mathfrak{D}$-based Piggyback Duality Theorem)

Let $\underline{\mathbf{M}}$ be a $\mathfrak{D}$-based total structure with reduct $\underline{\mathbf{M}}^\flat$ in $\mathfrak{D}$. Then an alter ego $\underset{\sim}{\mathbf{M}}$ of $\underline{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$ provided that there is a finite subset Ω of $\mathfrak{D}(\underline{\mathbf{M}}^\flat, \underline{\mathbf{D}})$ such that

(0) *each $\omega \in \Omega$ is continuous with respect to the topologies on $\underset{\sim}{\mathbf{M}}$ and $\underset{\sim}{\mathbf{D}}$,*

(1) $\Omega \circ \mathrm{Clo}_1(\underset{\sim}{\mathbf{M}})$ *separates the points of M, and*

(2) $\underset{\sim}{\mathbf{M}}$ *entails every relation in* $\max_{\underline{\mathbf{M}}} \Omega^{-1}(\leqslant)$.

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Remark 1

When $\underline{\mathbf{M}}$ is finite, this gives us a recipe for $\underset{\sim}{\mathbf{M}} = \langle M; G_2, R_2, \mathcal{T}\rangle$:

- choose $G_2 = \mathrm{End}(\underline{\mathbf{M}})$,
- choose $\Omega \subseteq \mathcal{D}(\underline{\mathbf{M}}^\flat, \underline{\mathbf{D}})$ so that $\Omega \circ \mathrm{End}(\underline{\mathbf{M}})$ separates the points of M. (The choice $\Omega = \mathcal{D}(\underline{\mathbf{M}}^\flat, \underline{\mathbf{D}})$ always works.)
- choose $R_2 = \max_{\underline{\mathbf{M}}} \Omega^{-1}(\leqslant)$.

Remarks continued

Assume that $\underline{\mathbf{M}}$ is finite.

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The theorem gives us a recipe for $\underset{\sim}{\mathbf{M}} = \langle M; G_2, R_2, \mathcal{T}\rangle$:

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- choose $R_2 = \max_{\underline{\mathbf{M}}} \Omega^{-1}(\leqslant)$.

Remark 2

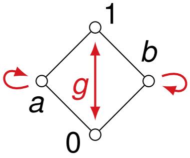
- Choose $\Omega = \mathcal{D}(\underline{\mathbf{M}}^\flat, \underline{\mathbf{D}})$. Then $\underset{\sim}{\mathbf{M}} := \langle M; \max_{\underline{\mathbf{M}}} \Omega^{-1}(\leqslant), \mathcal{T}\rangle$ yields a duality on $\mathcal{A} = \mathrm{ISP}(\underline{\mathbf{M}})$.
- To minimise the size of $R_2 = \max_{\underline{\mathbf{M}}} \Omega^{-1}(\leqslant)$ we should use $G_2 = \mathrm{End}(\underline{\mathbf{M}})$ and choose $\Omega \subseteq \mathcal{D}(\underline{\mathbf{M}}^\flat, \underline{\mathbf{D}})$ as small as possible.

Outline

De Morgan algebras again

A natural duality for De Morgan algebras

The class of De Morgan Algebras equals $\mathcal{A} = \mathrm{ISP}(\underline{\mathbf{M}})$, where $\underline{\mathbf{M}} = \langle \{0, a, b, 1\}; \vee, \wedge, g, 0, 1 \rangle$:



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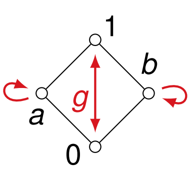

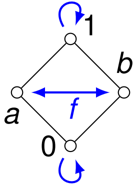

Choose f to be the automorphism of $\underline{\mathbf{M}}$ shown on the right and let $\omega \colon \underline{\mathbf{M}}^\flat \to \underline{\mathbf{D}}$ be the map with kernel $\{0, a \mid b, 1\}$.

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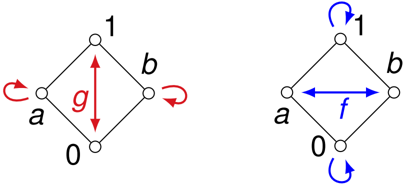

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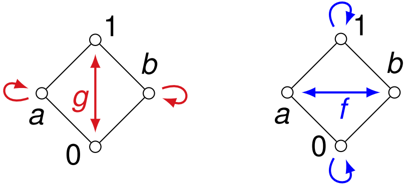

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It is clear that $\omega \circ \{\mathrm{id}_M, f\}$ separates the points of M.

It remains to calculate $\max_{\underline{\mathbf{M}}}\{\omega\}^{-1}(\leqslant)$, i.e., the maximal De Morgan subuniverses of

$$(\omega, \omega)^{-1}(\leqslant) = \{\, (a, b) \in M^2 \mid \omega(a) \leqslant \omega(b) \,\}.$$

De Morgan algebras again

A very useful result

Lemma

Let $\mathbf{A} = \langle A; \vee, \wedge, g, 0, 1\rangle$, *where* g *is either*

- *an endomorphism of* $\mathbf{A}^\flat$, *or*
- *a dual-endomorphism of* $\mathbf{A}^\flat$.

Let $\mathbf{L}$ *be a* $\{0, 1\}$*-sublattice of* $\mathbf{A}^\flat$, *then there is a largest subuniverse* L° *of* $\mathbf{A}$ *satisfying* $L^\circ \subseteq L$.

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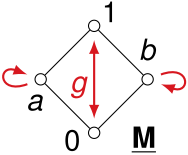
Since $(\omega, \omega)^{-1}(\leqslant) := \{\, (a, b) \in M^2 \mid \omega(a) \leqslant \omega(b) \,\}$ is a $\{0, 1\}$-sublattice of $\underline{\mathbf{M}}^\flat$, it follows from the lemma that there is a unique De Morgan subuniverse of $\underline{\mathbf{M}}^2$ that is maximal in $(\omega, \omega)^{-1}(\leqslant)$, namely $(\omega, \omega)^{-1}(\leqslant)^\circ$.

De Morgan algebras again

Calculating $\max_{\underline{\mathbf{M}}}\{\omega\}^{-1}(\leqslant)$



$\omega\colon \underline{\mathbf{M}}^\flat \to \underline{\mathbf{D}}$ is the map with kernel $\{0, a \mid b, 1\}$. Hence

$$\begin{aligned}(\omega,\omega)^{-1}(\leqslant) &= \{0, a\} \times \{0, a, b, 1\} \cup \{b, 1\} \times \{b, 1\} \\ &= M^2 \backslash \{b0, ba, 10, 1a\} \\ &= \{00, 0a, 0b, 01, a0, aa, ab, a1, bb, b1, 1b, 11\}\end{aligned}$$

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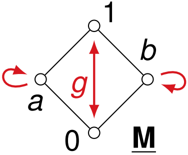

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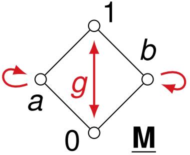

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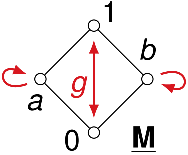

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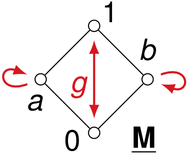

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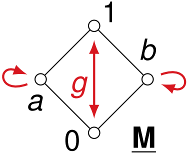

$\omega\colon \underline{\mathbf{M}}^\flat \to \underline{\mathbf{D}}$ is the map with kernel $\{0, a \mid b, 1\}$. Hence

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\begin{aligned}
(\omega,\omega)^{-1}(\leqslant) &= \{0,a\}\times\{0,a,b,1\} \cup \{b,1\}\times\{b,1\}\\
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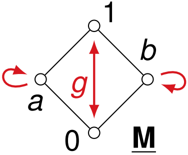

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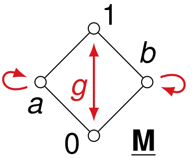

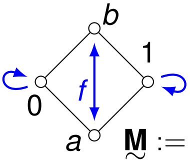

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De Morgan algebras again

Theorem (Cornish and Fowler)

$\underset{\sim}{\mathbf{M}} := \langle M; f, \preccurlyeq, \mathcal{T} \rangle$ *yields a duality on the class* $\mathcal{A} = \mathbb{ISP}(\underline{\mathbf{M}})$ *of De Morgan algebras.*

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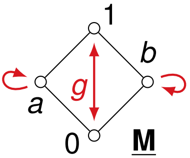

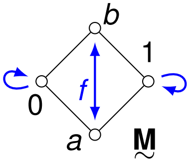

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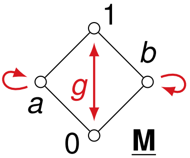

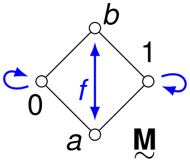

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- The dual category $\mathcal{X} = \mathrm{IS_cP^+}(\underset{\sim}{\mathbf{M}})$ is the category of Priestley spaces with a continuous order-reversing map f satisfying $f^2 = \mathrm{id}$.

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- Surprisingly, this is precisely the same as the restricted Priestley duality for De Morgan algebras. (More on this later.)

Natural dualities for Gödel algebras of degree n

Let $\underline{\mathbf{C}}_n = \langle C_n; \vee, \wedge, \to, 0, 1\rangle$ be the n-element chain regarded as a Heyting algebra. Thus,

$$a \to b = \begin{cases} 1 & \text{if } a \leqslant b, \\ 0 & \text{if } a > b. \end{cases}$$

The class $\mathcal{G}_n := \mathrm{ISP}(\underline{\mathbf{C}}_n)$ is the class of Gödel algebras of degree n.

- Define $\omega \colon \underline{\mathbf{C}}_n^\flat \to \underline{\mathbf{D}}$ by $\omega = \chi_{\{1\}}$.

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- For all $a < b$ in $\underline{\mathbf{C}}_n$, there exists $u \in \mathrm{End}(\underline{\mathbf{C}}_n)$ with $u(a) < u(b) = 1$.
- Hence $\omega \circ \mathrm{End}(\underline{\mathbf{C}}_n)$ separates the points of C_n.

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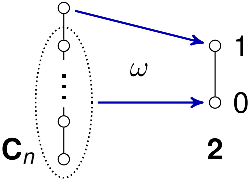
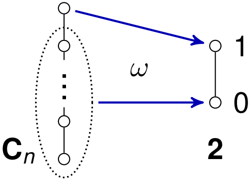
- Define $\omega \colon \underline{\mathbf{C}}_n^\flat \to \underline{\mathbf{D}}$ by $\omega = \chi_{\{1\}}$.
- For all $a < b$ in $\underline{\mathbf{C}}_n$, there exists $u \in \mathrm{End}(\underline{\mathbf{C}}_n)$ with $u(a) < u(b) = 1$.
- Hence $\omega \circ \mathrm{End}(\underline{\mathbf{C}}_n)$ separates the points of C_n.
- Thus $\underset{\sim}{\mathbf{C}}_n := \langle C_n; \mathrm{End}(\underline{\mathbf{C}}_n), \max_{\underline{\mathbf{C}}_n}\{\omega\}^{-1}(\leqslant), \mathcal{T}\rangle$ yields a duality on $\mathcal{G}_n$.

Natural dualities for Gödel algebras of degree n



Now let $\mathbf{r} \leqslant \underline{\mathbf{C}}_n^2$ with $r \subseteq (\omega, \omega)^{-1}(\leqslant)$.

Natural dualities for Gödel algebras of degree n



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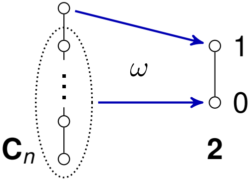

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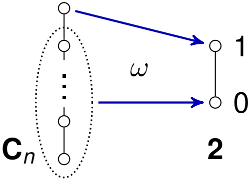

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$$\begin{aligned} (a,b),(a,c) \in r &\implies (1, b \to c) = (a,b) \to (a,c) \in r \\ &\implies 1 = \omega(1) \leqslant \omega(b \to c) \\ &\implies \omega(b \to c) = 1 \\ &\implies b \to c = 1 \\ &\implies b \leqslant c \quad \text{and } c \leqslant b \text{, by symmetry} \\ &\implies b = c \end{aligned}$$

Hence r is the graph of a partial endomorphism of $\underline{\mathbf{C}}_n$.

Natural dualities for Gödel algebras

We now know that $\underset{\sim}{\mathbf{C}}_n := \langle C_n; \mathrm{End}(\underline{\mathbf{C}}_n), H, \mathcal{T}\rangle$ yields a duality on $\mathcal{G}_n$, where H is the set of proper partial endomorphisms of $\underline{\mathbf{C}}_n$.

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In fact, H can be removed without destroying the duality, i.e.,

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In fact, H can be removed without destroying the duality, i.e.,

$$\mathrm{End}(\underline{\mathbf{C}}_n) \text{ entails every } h \in H.$$

The shortest proof of this uses three straightforward general results.

Three easy exercises for you

Let $\underline{\mathbf{M}}$ be a finite algebra.

Corollary 8.1.4

An alter ego $\underset{\sim}{\mathbf{M}}$ of $\underline{\mathbf{M}}$ entails an algebraic relation s provided $\underset{\sim}{\mathbf{M}}$ yields a duality on some isomorphic copy of the algebra $\mathbf{s}$.

Exercise 2.3

Assume that $\underset{\sim}{\mathbf{M}}$ is an alter ego of $\underline{\mathbf{M}}$ that yields a duality on $\mathbf{A} \in \mathsf{ISP}(\underline{\mathbf{M}})$, then $\underset{\sim}{\mathbf{M}}$ yields a duality on every retract of $\mathbf{A}$.

Exercise 2.4

$\underset{\sim}{\mathbf{M}} = \langle M; \mathrm{End}(\underline{\mathbf{M}}), \mathcal{T}\rangle$ yields a duality on the algebra $\underline{\mathbf{M}}$.

Time permitting, I will explain how it follows easily from these results that $\underset{\sim}{\mathbf{C}}_n = \langle C_n; \mathrm{End}(\underline{\mathbf{M}}), \mathcal{T}\rangle$ entails every partial endomorphism of $\underline{\mathbf{C}}_n$.

Natural dualities for Gödel algebras of degree n

Theorem (Davey, a long time ago)

$\underset{\sim}{\mathbf{C}}_n := \langle C_n; \mathrm{End}(\underline{\mathbf{C}}_n), \mathcal{T}\rangle$ *yields a duality on the class* $\mathcal{G}_n$ *of Gödel algebras of degree n.*

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- For $n \geqslant 4$, this duality is not strong.
- It can be made strong by adding back the partial endomorphisms.
- It provides a good example of when a non-full duality might be easier to use that a full or strong duality.

Outline

$\mathcal{D}$-based Piggyback Strong Duality Theorem

Notation

- Denote the functors that give the Priestley duality between the category $\mathcal{D} = \mathsf{ISP}(\underline{\mathbf{D}})$ of bounded distributive lattices and the category $\mathcal{P} = \mathsf{IS_cP^+}(\underset{\sim}{\mathbf{D}})$ of Priestley spaces by

$$H\colon \mathcal{D} \to \mathcal{P} \;\text{ and }\; K\colon \mathcal{P} \to \mathcal{D}.$$

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$$H\colon \mathcal{D} \to \mathcal{P} \quad \text{and} \quad K\colon \mathcal{P} \to \mathcal{D}.$$

- Thus, on objects, we have

$$H(\mathbf{A}) = \mathcal{D}(\mathbf{A}, \underline{\mathbf{D}}) \quad \text{and} \quad K(\mathbf{X}) = \mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{D}}),$$

for all $\mathbf{A} \in \mathcal{D}$ and $\mathbf{X} \in \mathcal{P}$.

$\mathcal{D}$-based Piggyback Strong Duality Theorem

Theorem (Davey, Haviar, Priestley 2015)

Let $\underline{\mathbf{M}}$ *be a total structure with reduct* $\underline{\mathbf{M}}^\flat$ *in* $\mathcal{D}$*, let* $\underset{\sim}{\mathbf{M}}$ *be an alter ego of* $\underline{\mathbf{M}}$ *and define* $\mathcal{A} := \mathsf{ISP}(\underline{\mathbf{M}})$ *and* $\mathfrak{X} := \mathsf{IS_cP^+}(\underset{\sim}{\mathbf{M}})$*. Assume that the structure on* $\underset{\sim}{\mathbf{M}}$ *includes an order relation* $\leqslant$ *such that* $\underset{\sim}{\mathbf{M}}^\flat := \langle M; \leqslant, \mathcal{T}\rangle$ *is a Priestley space, and there exists* $\omega \in \mathcal{D}(\underline{\mathbf{M}}^\flat, \underline{\mathbf{D}}) \cap \mathcal{P}(\underset{\sim}{\mathbf{M}}^\flat, \underset{\sim}{\mathbf{D}})$ *such that*

(1) $\omega \circ \mathsf{Clo}_1(\underset{\sim}{\mathbf{M}})$ *separates the points of* M*,*

(2) $\underset{\sim}{\mathbf{M}}$ *entails each* (*binary*) *relation in* $\max_{\underline{\mathbf{M}}}\{\omega\}^{-1}(\leqslant)$*, and*

(3) *if* $x \nleqslant y$ *in* $\underset{\sim}{\mathbf{M}}^\flat$*, then there exists* $t \in \mathsf{Clo}_1(\underline{\mathbf{M}})$ *such that* $\omega(t(x)) = 1$ *and* $\omega(t(y)) = 0$*.*

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Then

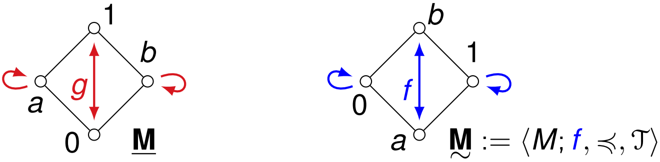
(a) $\underset{\sim}{\mathbf{M}}$ *fully dualises* $\underline{\mathbf{M}}$*,*

(b) $\underline{\mathbf{M}}$ *is injective in* $\mathcal{A}$ *and* $\underset{\sim}{\mathbf{M}}$ *is injective in* $\mathcal{X}$*, and*

(c) $D(\mathbf{A})^\flat \cong H(\mathbf{A}^\flat)$ *and* $E(\mathbf{X})^\flat \cong K(\mathbf{X}^\flat)$*, for all* $\mathbf{A} \in \mathcal{A}$*,* $\mathbf{X} \in \mathcal{X}$*.*

Outline

De Morgan algebras yet again



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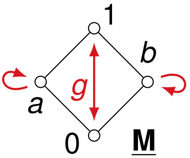
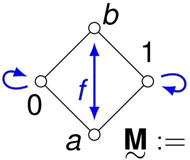
De Morgan algebras yet again





$\underset{\sim}{\mathbf{M}} := \langle M; f, \preccurlyeq, \mathcal{T} \rangle$

$\omega \colon \underline{\mathbf{M}}^\flat \to \underline{\mathbf{D}}$ is the map with kernel $\{0, a \mid b, 1\}$.

We already know that $\underset{\sim}{\mathbf{M}}$ satisfies conditions (1) and (2) of the $\mathcal{D}$-based Piggyback Strong Duality Theorem.

(3) if $x \not\preccurlyeq y$ in $\underset{\sim}{\mathbf{M}}^\flat$, then there exists $t \in \mathrm{Clo}_1(\underline{\mathbf{M}})$ such that $\omega(t(x)) = 1$ and $\omega(t(y)) = 0$.

There are 7 pairs to check. Some examples:

$$b \not\preccurlyeq a: \quad \omega(b) = 1 \;\;\&\;\; \omega(a) = 0$$

$$b \not\preccurlyeq 1: \quad \omega(g(b)) = 1 \;\;\&\;\; \omega(g(1)) = 0$$

$$0 \not\preccurlyeq 1: \quad \omega(g(0)) = 1 \;\;\&\;\; \omega(g(1)) = 0$$

De Morgan algebras yet again

Theorem (Cornish and Fowler)

- $\utilde{\mathbf{M}} := \langle M; f, \preccurlyeq, \mathcal{T} \rangle$ *yields a strong duality between the category* $\mathcal{A} = \mathrm{ISP}(\underline{\mathbf{M}})$ *of De Morgan algebras and the category* $\mathfrak{X} = \mathrm{IS_cP^+}(\utilde{\mathbf{M}})$.

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Theorem (Cornish and Fowler)

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- *The underlying ordered space of the natural dual is the Priestley dual:*

$$D(\mathbf{A})^\flat \cong H(\mathbf{A}^\flat) \text{ and } E(\mathbf{X})^\flat \cong K(\mathbf{X}^\flat),$$

for all $\mathbf{A} \in \mathcal{A}$ *and all* $\mathbf{X} \in \mathcal{X}$.

Ockham algebras

$\mathbf{A} = \langle A; \vee, \wedge, g, 0, 1\rangle$ is an Ockham algebra if $\mathbf{A}^\flat \in \mathcal{D}$ and g is a lattice-dual endomorphism of $\mathbf{A}^\flat$. We denote the equational class of Ockham algebras by $\mathcal{O}$.

- Let $\gamma\colon \mathbb{N}_0 \to \mathbb{N}_0$ be the successor function: $\gamma(n) := n + 1$ and let c denote Boolean complementation on $\{0, 1\}$.
- Define $\underline{\mathbf{M}}_1 := \langle \{0, 1\}^{\mathbb{N}_0} \mid \vee, \wedge, g, \underline{0}, \underline{1}\rangle$, where
  - $\vee$ and $\wedge$ are defined pointwise, $\underline{0}$ and $\underline{1}$ are the constant maps onto 0 and 1, respectively, and,
  - for all $a \in \{0, 1\}^{\mathbb{N}_0}$ we have $g(a) := c \circ a \circ \gamma$. Thus, g is given by shift left and then negate; for example,

$$g(0110010\dots) = (001101\dots).$$

Then $\underline{\mathbf{M}}_1$ is an Ockham algebra. Moreover, $\mathcal{O} = \mathsf{ISP}(\underline{\mathbf{M}}_1)$.

Piggyback duality for Ockham algebras

An alter ego $\underset{\sim}{\mathbf{M}}_2^{\mathcal{T}} = \langle \{0,1\}^{\mathbb{N}_0}; u, \preccurlyeq, \mathcal{T} \rangle$

- $u\colon \{0,1\}^{\mathbb{N}_0} \to \{0,1\}^{\mathbb{N}_0}$ is the left shift operator, given by $u(a) := a \circ \gamma$. Thus, for example,

  $$u(0110010\dots) = (110010\dots).$$

  Then $u \in \mathrm{End}(\underset{\sim}{\mathbf{M}}_1)$.
- $\preccurlyeq$ is the alternating order on $\{0,1\}^{\mathbb{N}_0}$, that is, for all $a, b \in \{0,1\}^{\mathbb{N}_0}$,

  $$a \preccurlyeq b \iff a(0) \leqslant b(0) \;\&\; a(1) \geqslant b(1) \;\&\; a(2) \leqslant b(2) \;\&\; \cdots.$$
- $\mathcal{T}$ is the product topology on $\{0,1\}^{\mathbb{N}_0}$ coming from the discrete topology on $\{0,1\}$.

Piggyback duality for Ockham algebras

Let $\omega := \pi_0 \colon \{0,1\}^{\mathbb{N}_0} \to \{0,1\}$.

Then, $\pi_0 \in \mathcal{D}(\underline{\mathbf{M}}_1^\flat, \underline{\mathbf{D}}) \cap \mathcal{P}((\underline{\mathbf{M}}_2^\flat)^{\mathcal{T}}, \underset{\sim}{\underline{\mathbf{D}}})$.

We now check Conditions (1)–(3) of the $\mathcal{D}$-based Piggyback Strong Duality Theorem.

(1) The set $\pi_0 \circ \mathrm{Clo}_1(\underline{\mathbf{M}}_2)$ separates the points of $M = \{0,1\}^{\mathbb{N}_0}$: indeed, let $a, b \in \{0,1\}^{\mathbb{N}_0}$ with $a \neq b$, then

$$\begin{aligned} & a \neq b \\ \Longrightarrow\ & (\exists n \in \mathbb{N}_0)\, a(n) \neq b(n) \\ \Longrightarrow\ & (\exists n \in \mathbb{N}_0)\, u^n(a)(0) = (a \circ \gamma^n)(0) \neq (b \circ \gamma^n)(0) = u^n(b)(0) \\ \Longrightarrow\ & (\exists n \in \mathbb{N}_0)\, (\pi_0 \circ u^n)(a) \neq (\pi_0 \circ u^n)(b). \end{aligned}$$

As u is in $\mathrm{Clo}_1(\underline{\mathbf{M}}_2)$, so is u^n. Hence $\pi_0 \circ \mathrm{Clo}_1(\underline{\mathbf{M}}_2)$ separates the points of M, that is, Condition (1) holds.

Piggyback duality for Ockham algebras

(2) We must find the binary relations r on M which form substructures of $\underline{\mathbf{M}}_1^2$ that are maximal in $(\pi_0, \pi_0)^{-1}(\leqslant)$. We have

$$(\pi_0, \pi_0)^{-1}(\leqslant) = \{\, (a, b) \in (\{0, 1\}^{\mathbb{N}_0})^2 \mid a(0) \leqslant b(0)\,\}.$$

Let $\mathbf{r}$ be a subalgebra of $\underline{\mathbf{M}}_1^2$ with $r \subseteq (\pi_0, \pi_0)^{-1}(\leqslant)$. Then

$$\begin{aligned}
(a, b) \in r \implies & (\forall n \in \mathbb{N}_0)\, (g^n(a), g^n(b)) \in r \\
\implies & (\forall n \in \mathbb{N}_0)\, g^n(a)(0) \leqslant g^n(b)(0) \\
\implies & a(0) \leqslant b(0) \ \ \&\ \ a(1) \geqslant b(1) \ \ \&\ \ a(2) \leqslant b(2) \ \ \&\ \ \cdots \\
\iff & a \preccurlyeq b.
\end{aligned}$$

Thus $r \subseteq \preccurlyeq$. Since $\preccurlyeq$ forms a subalgebra of $\underline{\mathbf{M}}_1^2$ and $\preccurlyeq \subseteq (\pi_0, \pi_0)^{-1}(\leqslant)$, it follows that $(\pi_0, \pi_0)^{-1}(\leqslant)^\circ = \preccurlyeq$.

Piggyback duality for Ockham algebras

(3) We must prove that $\pi_0 \circ \mathrm{Clo}_1(\underline{\mathbf{M}}_1)$ separates the relation $\preccurlyeq$, that is, if $a \not\preccurlyeq b$ in $\underline{\mathbf{M}}_2^\flat$, then there exists $t \in \mathrm{Clo}_1(\underline{\mathbf{M}}_1)$ such that $\pi_0(t(a)) = 1$ and $\pi_0(t(b)) = 0$. We have

$$\begin{aligned} & a \not\preccurlyeq b \text{ in } \underline{\mathbf{M}}_2^\flat \\ \iff\ & (\exists n \in \mathbb{N}_0) \begin{cases} a(n) = 1 \ \ \&\ \ b(n) = 0, & \text{if } n \text{ is even} \\ a(n) = 0 \ \ \&\ \ b(n) = 1, & \text{if } n \text{ is odd} \end{cases} \\ \implies\ & (\exists n \in \mathbb{N}_0)\ g^n(a)(0) = 1 \ \ \&\ \ g^n(b)(0) = 0 \\ \implies\ & (\exists n \in \mathbb{N}_0)\ \pi_0(g^n(a)) = 1 \ \ \&\ \ \pi_0(g^n(b)) = 0 \end{aligned}$$

as required, with $t(v) := g^n(v)$.

Piggyback duality for Ockham algebras

Theorem (Goldberg 1981/1983)

Let $\underline{\mathbf{M}}_1 := \langle \{0,1\}^{\mathbb{N}_0} \mid \vee, \wedge, g, \underline{0}, \underline{1} \rangle$ *and* $\underline{\mathbf{M}}_2^{\mathcal{T}} = \langle \{0,1\}^{\mathbb{N}_0}; u, \preccurlyeq, \mathcal{T} \rangle$.

- $\underline{\mathbf{M}}_2^{\mathcal{T}}$ *yields a strong duality between the category* $\mathcal{O} = \mathsf{ISP}(\underline{\mathbf{M}}_1)$ *of Ockham algebras and the category* $\mathcal{Y} = \mathsf{IS_cP^+}(\underset{\sim}{\mathbf{M}}_2^{\mathcal{T}})$ *of Ockham spaces.*

Piggyback duality for Ockham algebras

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- *The underlying ordered space of the natural dual is the Priestley dual:*

$$D(\mathbf{A})^\flat \cong H(\mathbf{A}^\flat) \quad \text{and} \quad E(\mathbf{X})^\flat \cong K(\mathbf{X}^\flat),$$

for all $\mathbf{A} \in \mathcal{A}$ *and all* $\mathbf{X} \in \mathcal{X}$.

Two-for-one piggyback duality for Ockham algebras

We get a second strong duality for free simply by swapping the topology from one structure to the other.

Theorem (Davey, Haviar, Priestley 2015)

Let $\underline{\mathbf{M}}_1 := \langle \{0,1\}^{\mathbb{N}_0} \mid \vee, \wedge, g, \underline{0}, \underline{1} \rangle$ *and* $\underline{\mathbf{M}}_2 = \langle \{0,1\}^{\mathbb{N}_0}; u, \preccurlyeq \rangle$.

Then

- $\underline{\mathbf{M}}_2^{\mathcal{T}} := \langle \{0,1\}^{\mathbb{N}_0}; u, \preccurlyeq, \mathcal{T} \rangle$ *strongly dualises* $\underline{\mathbf{M}}_1$*, and*
- $\underline{\mathbf{M}}_1^{\mathcal{T}} := \langle \{0,1\}^{\mathbb{N}_0}; \vee, \wedge, \neg, \underline{0}, \underline{1}, \mathcal{T} \rangle$ *strongly dualises* $\underline{\mathbf{M}}_2$*.*

Note that

- $\mathrm{ISP}(\underline{\mathbf{M}}_2)$ consists of ordered sets equipped with an order-reversing map, and
- $\mathrm{IS_cP^+}(\underline{\mathbf{M}}_1^{\mathcal{T}})$ consists of Boolean-topological Ockham algebras.

Outline

Some homework exercises for you

Some theory

- Prove the claims in Corollary 8.1.4, Exercise 2.3 and Exercise 2.4.

Some practice

In each case, use the useful lemma and compare your answer to the duality we have already found.

- Use the two-element set $\Omega = \mathfrak{D}(\underline{\mathbf{K}}^\flat, \underline{\mathbf{D}})$ to obtain a duality for Kleene algebras via the Piggyback Duality Theorem.
- Use the two-element set $\Omega = \mathfrak{D}(\underline{\mathbf{M}}^\flat, \underline{\mathbf{D}})$ to obtain a duality for De Morgan algebras via the Piggyback Duality Theorem.