Lecture 3: From dualities to full and strong dualities

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Outline

Natural dualities: the basics

A Natural duality for Kleene algebras

Full and strong dualities

The standard setup

- Let $\mathbf{\underline{M}}$ be a finite algebra let $\mathcal{A} := \mathbb{ISP}(\mathbf{\underline{M}})$ be the prevariety (= quasivariety) it generates.
- Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ be an alter ego of $\mathbf{\underline{M}}$, that is,
  - G is a set of operations on M, each of which is a homomorphism with respect to $\mathbf{\underline{M}}$,
  - H is a set of partial operations on M, each of which is a homomorphism with respect to $\mathbf{\underline{M}}$,
  - R is a set of relations on M, each of which is a subuniverse of the appropriate power of $\mathbf{\underline{M}}$, and
  - $\mathcal{T}$ is the discrete topology on M.
- Define $\mathcal{A} := \mathbb{ISP}(\mathbf{\underline{M}})$: the algebraic category of interest.
- Define $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$: the potential dual category for $\mathcal{A}$.

The standard setup

- The natural hom-functors $D\colon \mathcal{A} \to \mathcal{X}$ and $E\colon \mathcal{X} \to \mathcal{A}$ are defined by

  $$D(\mathbf{A}) := \mathcal{A}(\mathbf{A}, \mathbf{\underline{M}}) \leqslant \underset{\sim}{\mathbf{M}}^A \quad \text{and} \quad E(\mathbf{X}) := \mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}}) \leqslant \mathbf{\underline{M}}^X.$$
- For all $\mathbf{A} \in \mathcal{A}$, the naturally embedding

  $$e_{\mathbf{A}}\colon \mathbf{A} \to ED(\mathbf{A}) = \mathcal{X}(\mathcal{A}(\mathbf{A}, \mathbf{\underline{M}}), \underset{\sim}{\mathbf{M}})$$

  is defined by evaluation: $(\forall a \in A)\ e_{\mathbf{A}}(a)\colon \mathcal{A}(\mathbf{A}, \mathbf{\underline{M}}) \to \underset{\sim}{\mathbf{M}}$ is given by

  $$(\forall x \in \mathcal{A}(\mathbf{A}, \mathbf{\underline{M}}))\ \ e_{\mathbf{A}}(a)(x) := x(a)$$
- For all $\mathbf{X} \in \mathcal{X}$, the naturally embedding

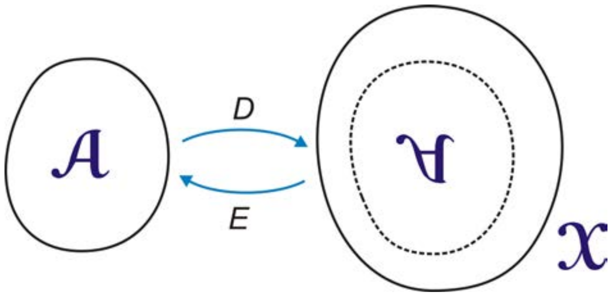
  $$\varepsilon_{\mathbf{X}}\colon \mathbf{X} \to DE(\mathbf{X}) = \mathcal{A}(\mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}}), \mathbf{\underline{M}})$$

  is defined by evaluation: $(\forall x \in X)\ \varepsilon_{\mathbf{X}}(x)\colon \mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}}) \to \mathbf{\underline{M}}$ is given by

  $$(\forall \alpha \in \mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}}))\ \varepsilon_{\mathbf{X}}(x)(\alpha) := \alpha(x).$$

Duality

If $e_{\mathbf{A}} \colon \mathbf{A} \to ED(\mathbf{A})$ is surjective and therefore an isomorphism, for all $\mathbf{A}$ in $\mathcal{A}$, then we say that $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$ (or that $\underset{\sim}{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$).



Theorem (2.2.7 Second Duality Theorem)

Assume that $\underset{\sim}{\mathbf{M}} = \langle M; G, R, \mathcal{T} \rangle$ is a total structure with R finite. If (IC) *holds, then $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$ and is injective in $\mathcal{X}$.*

Taming brute force with near unanimity

For $\ell \geqslant 1$, define $R_\ell := \mathrm{S}(\underline{\mathbf{M}}^\ell)$ and define $R_\omega := \bigcup_{\ell<\omega} R_\ell$.

Theorem (2.3.1 Brute Force Duality Theorem)

Brute force yields a duality on $\mathcal{A}_{\mathrm{fin}}$. Indeed, if $\underset{\sim}{\mathbf{M}} = \langle M; R_\omega, \mathcal{T} \rangle$, then (IC) *holds and therefore $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}_{\mathrm{fin}}$ and $\underset{\sim}{\mathbf{M}}$ is injective in $\mathcal{X}_{\mathrm{fin}}$.*

For $k \geqslant 2$, a $(k+1)$-ary term $n(v_1, \dots, v_{k+1})$ is called a near unanimity term or NU term for an algebra $\underline{\mathbf{M}}$ if $\underline{\mathbf{M}}$ satisfies

$$n(y, x, \dots, x) \approx n(x, y, x, \dots, x) \approx \cdots \approx n(x, \dots, x, y) \approx x.$$

Lemma (2.3.3 NU Lemma)

(*K. Baker and A. Pixley*) *Let $k \geq 2$ and assume that $\underline{\mathbf{M}}$ has a $(k+1)$-ary NU term. Let X be a subset of M^m and let $\alpha \colon X \to M$ be a map that preserves every relation in R_k. Then α preserves every relation in R_ω.*

The NU Duality Theorem

The following useful result is an immediate corollary.

Theorem (NU Duality Theorem)

Assume that $\underline{\mathbf{M}}$ is a finite algebra that has a $(k+1)$-ary NU term. Then $\underset{\sim}{\mathbf{M}} := \langle M; R_k, \mathcal{T} \rangle$ yields a duality on $\mathcal{A}$ and is injective in $\mathcal{X}$.

Lattices have a ternary NU term, namely the median

$$m(x, y, z) := (x \wedge y) \vee (y \wedge z) \vee (z \wedge x).$$

Thus we obtain the most widely used result in the theory.

Theorem (Lattice-based Duality Theorem)

Let $\underline{\mathbf{M}}$ be a finite lattice-based algebra. Then $\underset{\sim}{\mathbf{M}} := \langle M; R_2, \mathcal{T} \rangle$ yields a duality on $\mathcal{A}$ and is injective in $\mathcal{X}$.

Priestley duality via the Lattice-based Duality Theorem

In Lecture 2 we saw how to obtain (half of) Priestley duality from the Second Duality Theorem. As an application of the Lattice-based Duality Theorem, it is almost immediate.

- $\underline{\mathbf{D}} = \langle \{0, 1\}; \vee, \wedge, 0, 1 \rangle$ and $\underset{\sim}{\mathbf{D}} = \langle \{0, 1\}; \leqslant, \mathcal{T} \rangle$.

Theorem (Half of Priestley duality)

$\underset{\sim}{\mathbf{D}}$ yields a duality on the class $\mathcal{D} := \mathrm{ISP}(\underline{\mathbf{D}})$ of bounded distributive lattices, i.e., $e_{\mathbf{A}} \colon \mathbf{A} \to ED(\mathbf{A})$ is an isomorphism, for all $\mathbf{A} \in \mathcal{D}$.

Priestley duality via the Lattice-based Duality Theorem

We must show that, for all $\mathbf{A} \in \mathcal{D}$, the evaluation maps

$$e_{\mathbf{A}}(a) \colon \mathcal{D}(\mathbf{A}, \underline{\mathbf{D}}) \to \{0, 1\},$$

for $a \in A$, are the only continuous order-preserving maps.

Proof.
Let $\alpha \colon \mathcal{D}(\mathbf{A}, \underline{\mathbf{D}}) \to \{0, 1\}$ be a continuous order-preserving map. [To prove: α is an evaluation map, $e_{\mathbf{A}}(a)$, for some $a \in A$.]

- By the Lattice-based Duality Theorem, $\underset{\sim}{\mathbf{D}}' := \langle \{0, 1\}; R_2, \mathcal{T} \rangle$ yields a duality on $\mathcal{D}$.
- So the evaluations $e_{\mathbf{A}}(a)$ are the only continuous maps from $\mathcal{D}(\mathbf{A}, \underline{\mathbf{D}})$ to $\{0, 1\}$ that preserve the relations in R_2.
- Note that $R_2 = \{\Delta_{\{0,1\}}, \leqslant, \geqslant, \{0, 1\}^2\}$.
- But $\alpha \colon \mathcal{D}(\mathbf{A}, \underline{\mathbf{D}}) \to \{0, 1\}$ certainly preserves the trivial relations $\Delta_{\{0,1\}}$ and $\{0, 1\}^2$, and α preserves $\geqslant$ since it preserves $\leqslant$. Hence α preserve the four relations in R_2.
- Hence α is an evaluation, as $\underset{\sim}{\mathbf{D}}'$ yields a duality on $\mathcal{D}$. □

Refining an alter ego via entailment

Definition (Entailment)
Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T} \rangle$, let $\mathbf{A} \in \mathcal{A}$ and let s be an algebraic relation or (partial) operation on $\underline{\mathbf{M}}$.

- $G \cup H \cup R$ **entails** s **on** $D(\mathbf{A})$ if every continuous $G \cup H \cup R$-preserving map $\alpha \colon D(\mathbf{A}) \to M$ preserves s.
- $G \cup H \cup R$ **entails** s if $G \cup H \cup R$ entails s on $D(\mathbf{A})$ for all $\mathbf{A} \in \mathcal{A}$.

The following lemma is trivial but useful.

Lemma
Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T} \rangle$ and $\underset{\sim}{\mathbf{M}}' = \langle M; G', H', R', \mathcal{T} \rangle$ be alter egos of $\underline{\mathbf{M}}$. If $\underset{\sim}{\mathbf{M}}'$ yields a duality of $\mathcal{A}$ and $G \cup H \cup R$ entails s, for all $s \in G' \cup H' \cup R'$, then $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$.

Constructs for entailment

On pages 25–27 of *The Lonely Planet Guide to the Theory of Natural Dualities* there is a list of 15 constructs for entailment. Some are:

(1) **Trivial relations** If θ is an equivalence relation on $\{1, \dots, n\}$ then any $G \cup H \cup R$ entails the relation $\Delta^\theta := \{(c_1, \dots, c_n) \mid i\,\theta\,j \Rightarrow c_i = c_j\}$. Special cases are Δ_M and M^2.

(4) **Permutation** r entails $r^\sigma := \{(c_1, \dots, c_n) \mid (c_{\sigma(1)}, \dots, c_{\sigma(n)}) \in r\}$. Converse $r^{\smile} := \{(c_1, c_2) \mid (c_2, c_1) \in r\}$ is a special case.

(6) **Intersection** If r and s are n-ary, the $\{r, s\}$ entails $r \cap s$.

(7) **Product** $\{r, s\}$ entails $r \times s$.

N.B. A construct that is not allowed is the relational product $r \cdot s$ of two binary relations!

4.3.9 Natural duality for Kleene algebras

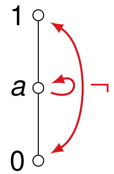
An algebra $\underline{\mathbf{K}} = \langle K; \vee, \wedge, \neg, 0, 1 \rangle$ is called a **Kleene algebra** if it is a bounded distributive lattice satisfying the axioms

$$\neg(x \wedge y) \approx \neg x \vee \neg y, \quad \neg(x \vee y) \approx \neg x \wedge \neg y, \quad \neg 0 \approx 1,$$

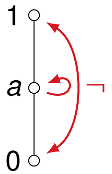
$$\neg\neg x \approx x, \quad x \wedge \neg x \leq y \vee \neg y.$$

The models of these axioms form a variety $\mathcal{K} = \mathrm{ISP}(\underline{\mathbf{K}})$ generated by the three-element chain

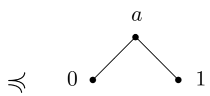
$$\underline{\mathbf{K}} = \langle \{0, a, 1\}; \vee, \wedge, \neg, 0, 1 \rangle :$$



4.3.9 Natural duality for Kleene algebras



- By the Lattice-based Duality Theorem, $\underset{\sim}{\mathbf{K}}' := \langle \{0, a, 1\}; R_2, \mathcal{T} \rangle$ yields a duality on $\mathcal{K}$.
- We must find all subuniverses of $\underline{\mathbf{K}}^2$.
- Let $K_0 = \{0, 1\}$, let $\preccurlyeq = \{00, aa, 11, 0a, 1a\}$ and let $\sim = K^2 \backslash \{01, 10\}$.



4.3.9 Natural duality for Kleene algebras

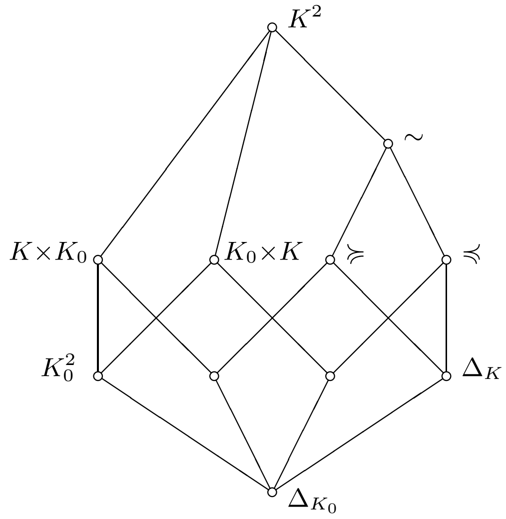


Figure: 8.1 The lattice $\langle R_2; \subseteq \rangle$ of subuniverses of $\underline{\mathbf{K}}^2$

4.3.9 Natural duality for Kleene algebras

Let $R = \{K_0, \preccurlyeq, \sim\}$. Then R entails every relation in R_2 since

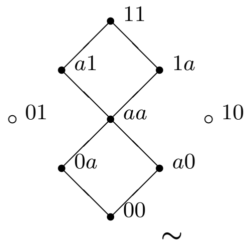
- R entails the trivial relation K, whence R entails the products $K \times K_0$, $K_0 \times K$ and $K \times K$,
- R entails the converse $\succcurlyeq$ of $\preccurlyeq$,
- and of course R entails $\sim$ (as $\sim \in R$).

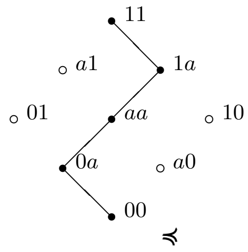
Thus R entails every meet-irreducible relation in the lattice $\langle R_2; \subseteq \rangle$ and so entails every relation in R_2 via intersection.

Theorem (Part of 4.3.10)

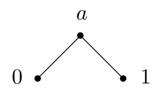
$\underset{\sim}{\mathbf{K}} = \langle K; K_0, \preccurlyeq, \sim, \mathcal{T} \rangle$ *yields a duality on the class $\mathcal{K}$ of Kleene algebras.*

4.3.9 Natural duality for Kleene algebras





- The uncertainty order on $\{0, a, 1\}$:



- Note that $\sim = \preccurlyeq \cdot \succcurlyeq$
- We will now see that removing $\sim$ will destroy the duality.
- In fact, the duality is optimal.

8.1.3 The Test Algebra Lemma

- Our claim is that, while $\underset{\sim}{\mathbf{K}} = \langle K; K_0, \preccurlyeq, \sim, \mathcal{T}\rangle$ yields a duality on the class $\mathcal{K}$ of Kleene algebras, the alter ego $\underset{\sim}{\mathbf{K}}^* = \langle K; K_0, \preccurlyeq, \mathcal{T}\rangle$ does not.
- To prove this, we must find an algebra $\mathbf{A} \in \mathcal{K}$ and a continuous map $\gamma\colon \mathcal{K}(\mathbf{A}, \underline{\mathbf{K}}) \to K$ that preserves K_0 and $\preccurlyeq$ but is not an evaluation,
- or equivalently, $\{K_0, \preccurlyeq\}$ does not entail $\sim$ on $\mathcal{K}(\mathbf{A}, \underline{\mathbf{K}})$.

In fact, there is a canonical choice for $\mathbf{A}$.

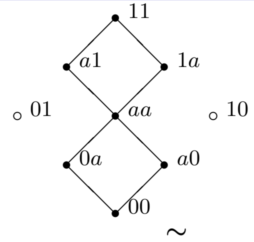
Lemma (Test Algebra Lemma)

Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ and let s be an algebraic relation or (partial) operation on $\underline{\mathbf{M}}$ and let $\mathbf{s}$ be the corresponding subalgebra of $\underline{\mathbf{M}}^n$. Then the following are equivalent:

(i) *$G \cup H \cup R$ entails s;*

(ii) *$G \cup H \cup R$ entails s on $D(\mathbf{s})$.*

Back to the relation $\sim$

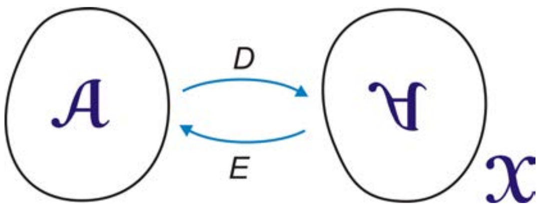


$D(\sim) = \mathcal{K}(\sim, \underline{\mathbf{K}}) = \{\rho_1, \rho_2\}$,
where $\rho_i\colon \sim \to \underline{\mathbf{K}}$,
for $i \in \{1, 2\}$,
are the two projections.

- Define $\gamma\colon D(\sim) \to K$ by $\gamma(\rho_1) = 0$ and $\gamma(\rho_2) = 1$.
- It is trivial that $(\rho_1, \rho_2) \in \sim^{D(\sim)}$.
- But $(\gamma(\rho_1), \gamma(\rho_2)) = (0, 1) \notin \sim$
- Hence the map γ does not preserve $\sim$.
- $K_0^{D(\sim)} = \varnothing$, as $\rho_i(a, a) = a \notin K_0$, and
- $\preccurlyeq^{D(\sim)} = \{(\rho_1, \rho_1), (\rho_2, \rho_2)\} = \Delta_{D(\sim)}$.
- Hence the map γ preserves both K_0 and $\preccurlyeq$.
- Thus $\{K_0, \preccurlyeq\}$ does not entail $\sim$. $\square$

Full Duality

If $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$ and , in addition, $\varepsilon_{\mathbf{X}}\colon \mathbf{X} \to DE(\mathbf{X})$ is a surjection and therefore an isomorphism, for all $\mathbf{X}$ in $\mathcal{X}$, then $\underset{\sim}{\mathbf{M}}$ yields a full duality on $\mathcal{A}$ (or $\underset{\sim}{\mathbf{M}}$ fully dualises $\underline{\mathbf{M}}$).



Equivalently, $\underset{\sim}{\mathbf{M}}$ yields a full duality on $\mathcal{A}$ if the dual adjunction $\langle D, E, e, \varepsilon\rangle$ is a dual category equivalence between $\mathcal{A}$ and $\mathcal{X}$.

Strong duality

Let $\underset{\sim}{\mathbf{M}}$ be any alter ego of an algebra $\underline{\mathbf{M}}$, and let

$$D\colon \mathcal{A} \to \mathcal{X} \quad \text{and} \quad E\colon \mathcal{X} \to \mathcal{A}$$

be the induced hom-functors.

- $\underset{\sim}{\mathbf{M}}$ is injective in the category $\mathcal{X}$ if, for every embedding $\varphi\colon \mathbf{X} \rightarrowtail \mathbf{Y}$ and every morphism $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ in $\mathcal{X}$, there is a morphism $\beta\colon \mathbf{Y} \to \underset{\sim}{\mathbf{M}}$ such that $\beta \circ \varphi = \alpha$.

$$\begin{tikzcd} \mathbf{X} \arrow[r, tail, "\varphi"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circ"] & \mathbf{Y} \arrow[dl, dashed, "\beta"] \\ \underset{\sim}{\mathbf{M}} & \end{tikzcd}$$

Strong duality

If $\underset{\sim}{\mathbf{M}}$ fully dualises $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}$ is injective in $\mathcal{X}$ (so that surjections in $\mathcal{A}$ correspond to embeddings in $\mathcal{X}$), we say that $\underset{\sim}{\mathbf{M}}$ yields a strong duality on $\mathcal{A}$ (or that $\underset{\sim}{\mathbf{M}}$ strongly dualises $\underline{\mathbf{M}}$).

The CD Strong Duality Theorem

Let $\underline{\mathbf{M}}$ be a finite algebra.

- For all $\mathbf{N} \leqslant \underline{\mathbf{M}}$ define $\operatorname{irr}(\mathbf{N})$ to be the least ℓ such that $0_{\mathbf{N}}$ in $\operatorname{Con}(\mathbf{N})$ is a meet of ℓ meet-irreducible congruences.
- Define $\operatorname{Irr}(\underline{\mathbf{M}}) := \max\{\operatorname{irr}(\mathbf{N}) \mid \mathbf{N} \text{ is a subalgebra of } \underline{\mathbf{M}}\}$. $\operatorname{Irr}(\underline{\mathbf{M}})$ is called the irreducibility index of $\underline{\mathbf{M}}$.
- Define $C := \{a \in M \mid \{a\} \text{ is a subuniverse of } \underline{\mathbf{M}}\}$ regarded as a set of nullary operations on M.
- For all $n \geqslant 1$, define H_n to be the set of maps $h\colon D \to M$ such that $\mathbf{D}$ is a subalgebra of $\underline{\mathbf{M}}^n$ and h is a homorphism.

Theorem (3.3.7 CD Strong Duality Theorem)
Assume that $\underline{\mathbf{M}}$ is a finite algebra and that $\underset{\sim}{\mathbf{M}} := \langle M; R, \mathcal{T}\rangle$ dualises $\underline{\mathbf{M}}$. If $\operatorname{Var}(\underline{\mathbf{M}})$ is congruence distributive and $\operatorname{Irr}(\underline{\mathbf{M}}) = n$, then $\underset{\sim}{\mathbf{M}} := \langle M; C \cup H_n, R, \mathcal{T}\rangle$ strongly dualises $\underline{\mathbf{M}}$.

N.B. $\operatorname{Var}(\underline{\mathbf{M}})$ is congruence distributive if $\underline{\mathbf{M}}$ is lattice based.

Distributive lattices revisited

- $\underline{\mathbf{D}} = \langle\{0,1\}; \vee, \wedge, 0, 1\rangle$ and $\underset{\sim}{\mathbf{D}} = \langle\{0,1\}; \leqslant, \mathcal{T}\rangle$.

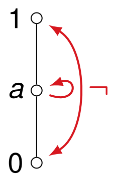
Theorem (Priestley duality is strong)
$\underset{\sim}{\mathbf{D}}$ yields a strong duality between the class $\mathcal{D} := \mathrm{ISP}(\underline{\mathbf{D}})$ of bounded distributive lattices and the class $\mathcal{P} = \mathrm{IS_cP^+}(\underset{\sim}{\mathbf{D}})$ of Priestley spaces, i.e., $\underset{\sim}{\mathbf{D}}$ is injective in $\mathcal{P}$ and, for all $\mathbf{A} \in \mathcal{D}$ and $\mathbf{X} \in \mathcal{P}$,

- *$e_{\mathbf{A}}\colon \mathbf{A} \to ED(\mathbf{A})$ and $\varepsilon_{\mathbf{X}}\colon \mathbf{X} \to ED(\mathbf{X})$ are isomorphisms.*

Proof.

- We already know that $\underset{\sim}{\mathbf{D}}$ yields a duality on $\mathcal{D}$.
- $\underline{\mathbf{D}}$ is simple and has no subalgebras, so $\operatorname{Irr}(\underline{\mathbf{D}}) = 1$.
- It follows from the CD Strong Duality Theorem that $\underset{\sim}{\mathbf{D}}' = \langle\{0,1\}; \mathrm{id}_D, \leqslant, \mathcal{T}\rangle$ yields a strong duality on $\mathcal{D}$.
- Clearly id_D can be removed without affecting the result.
- Hence $\underset{\sim}{\mathbf{D}}$ yields a strong duality on $\mathcal{D}$. □

Kleene algebras revisited



$\underline{\mathbf{K}} = \langle\{0, a, 1\}; \vee, \wedge, \neg, 0, 1\rangle$
and
$\underset{\sim}{\mathbf{K}} = \langle K; K_0, \preccurlyeq, \sim, \mathcal{T}\rangle$.

Theorem (Strong duality for Kleene algebras)
$\underset{\sim}{\mathbf{K}}$ yields a strong duality between the class $\mathcal{K} := \mathrm{ISP}(\underline{\mathbf{K}})$ of Kleene algebras and the class $\mathcal{X} = \mathrm{IS_cP^+}(\underset{\sim}{\mathbf{K}})$.

Proof.

- We already know that $\underset{\sim}{\mathbf{K}}$ yields a duality on $\mathcal{K}$.
- $\underline{\mathbf{K}}$ and $\underline{\mathbf{K}}_0$ are simple, so $\operatorname{Irr}(\underline{\mathbf{K}}) = 1$.
- It follows from the CD Strong Duality Theorem that $\underset{\sim}{\mathbf{K}}' = \langle K; \mathrm{id}_K, \mathrm{id}_{K_0}, K_0, \preccurlyeq, \sim, \mathcal{T}\rangle$ yields a strong duality on $\mathcal{K}$.
- id_K and id_{K_0} can be removed without affecting the result.
- Hence $\underset{\sim}{\mathbf{K}}$ yields a strong duality on $\mathcal{K}$. □

Partial operations can't be avoided

Theorem (6.1.2 Total Structure Theorem)
Assume that $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ yields a strong duality on $\mathcal{A}$. The following are equivalent:

(i) *some total structure $\underset{\sim}{\mathbf{M}}'$ yields a strong duality on $\mathcal{A}$;*
(ii) *for each natural number n, every n-ary partial operation $h \in H$ extends to a homomorphism $g\colon \underline{\mathbf{M}}^n \to \underline{\mathbf{M}}$;*
(iii) *$\underset{\sim}{\mathbf{M}}$ is injective in $\mathcal{A}$.*

Let $\underline{\mathbf{M}}$ be any finite lattice-based algebra that is not injective in $\mathcal{A} = \mathrm{ISP}(\underline{\mathbf{M}})$. Then

- there is an alter ego $\underset{\sim}{\mathbf{M}}$ that yields a strong duality on $\mathcal{A}$,
- but any such $\underset{\sim}{\mathbf{M}}$ must include partial operations in its type.

Further examples

Some exercises for you. Use the Lattice-based Duality Theorem and the CD Strong Duality Theorem to find a strong duality for $\mathcal{A} := \mathsf{ISP}(\underline{\mathbf{M}})$ in each of the following cases.
Is your duality optimal?

1. Median algebras. $\underline{\mathbf{M}} = \langle \{0, 1\}; m \rangle$, where $m\colon \{0, 1\}^3 \to \{0, 1\}$ is the median operation.
2. Stone algebras. $\underline{\mathbf{M}} = \langle \{0, a, 1\}; \vee, \wedge, {}^*, 0, 1 \rangle$, where $\langle \{0, a, 1\}; \vee, \wedge, 0, 1 \rangle$ is a chain with $0 < a < 1$ and * is given by $0^* = 1$ and $a^* = 1^* = 0$.
3. Double Stone algebras. $\underline{\mathbf{M}} = \langle \{0, a, b, 1\}; \vee, \wedge, {}^*, {}^+, 0, 1 \rangle$, where $\langle \{0, a, b, 1\}; \vee, \wedge, 0, 1 \rangle$ is a chain with $0 < a < b < 1$ and * and $^+$ are given by $0^* = 1$ and $a^* = b^* = 1^* = 0$, and $1^+ = 0$ and $0^+ = a^+ = b^+ = 1$.
4. 3-valued Gödel algebras. $\underline{\mathbf{M}} = \langle \{0, a, 1\}; \vee, \wedge, \to, 0, 1 \rangle$, where $\langle \{0, a, 1\}; \vee, \wedge, 0, 1 \rangle$ is a chain with $0 < a < 1$ and $x \to y = 1$, if $x \leqslant y$, and $x \to y = y$, if $x > y$.