Lecture 3: From dualities to full and strong dualities

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Outline

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The standard setup

- Let $\underline{\mathbf{M}}$ be a finite algebra let $\mathcal{A} := \mathsf{ISP}(\underline{\mathbf{M}})$ be the prevariety (= quasivariety) it generates.
- Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T} \rangle$ be an alter ego of $\underline{\mathbf{M}}$, that is,
  - G is a set of operations on M, each of which is a homomorphism with respect to $\underline{\mathbf{M}}$,
  - H is a set of partial operations on M, each of which is a homomorphism with respect to $\underline{\mathbf{M}}$,
  - R is a set of relations on M, each of which is a subuniverse of the appropriate power of $\underline{\mathbf{M}}$, and
  - $\mathcal{T}$ is the discrete topology on M.
- Define $\mathcal{A} := \mathsf{ISP}(\underline{\mathbf{M}})$: the algebraic category of interest.
- Define $\mathcal{X} := \mathsf{IS_cP^+}(\underset{\sim}{\mathbf{M}})$: the potential dual category for $\mathcal{A}$.

The standard setup

- The natural hom-functors $D\colon \mathcal{A} \to \mathfrak{X}$ and $E\colon \mathfrak{X} \to \mathcal{A}$ are defined by

$$D(\mathbf{A}) := \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}}) \leqslant \underset{\sim}{\mathbf{M}}^{A} \quad \text{and} \quad E(\mathbf{X}) := \mathfrak{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}}) \leqslant \underline{\mathbf{M}}^{X}.$$

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- For all $\mathbf{A} \in \mathcal{A}$, the naturally embedding

$$e_{\mathbf{A}}\colon \mathbf{A} \to ED(\mathbf{A}) = \mathfrak{X}(\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}}), \underset{\sim}{\mathbf{M}})$$

is defined by evaluation: $(\forall a \in A)\ e_{\mathbf{A}}(a)\colon \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}}) \to \underset{\sim}{\mathbf{M}}$ is given by

$$\big(\forall x \in \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})\big)\ \ e_{\mathbf{A}}(a)(x) := x(a)$$

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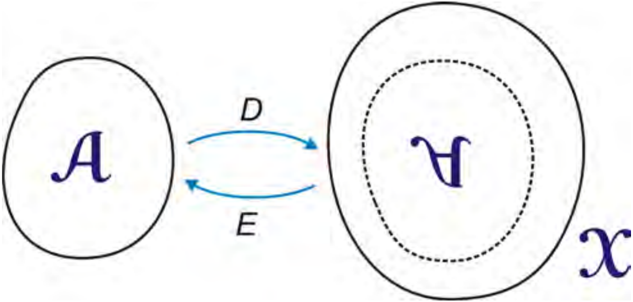
$$\big(\forall \alpha \in \mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}})\big)\ \varepsilon_{\mathbf{X}}(x)(\alpha) := \alpha(x).$$

Duality

If $e_{\mathbf{A}} \colon \mathbf{A} \to ED(\mathbf{A})$ is surjective and therefore an isomorphism, for all $\mathbf{A}$ in $\mathcal{A}$, then we say that $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$ (or that $\underset{\sim}{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$).



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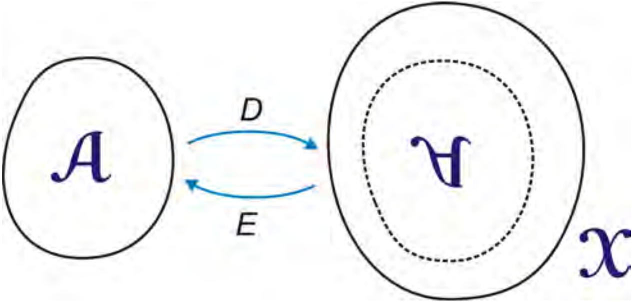

Theorem (2.2.7 Second Duality Theorem)

Assume that $\underset{\sim}{\mathbf{M}} = \langle M; G, R, \mathcal{T} \rangle$ is a total structure with R finite. If (IC) *holds, then $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$ and is injective in $\mathcal{X}$.*

Taming brute force with near unanimity

For $\ell \geqslant 1$, define $R_\ell := \mathsf{S}(\underline{\mathbf{M}}^\ell)$ and define $R_\omega := \bigcup_{\ell<\omega} R_\ell$.

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Theorem (2.3.1 Brute Force Duality Theorem)

Brute force yields a duality on $\mathcal{A}_{\mathrm{fin}}$. Indeed, if $\underset{\sim}{\mathbf{M}} = \langle M; R_\omega, \mathcal{T}\rangle$, then (IC) *holds and therefore $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}_{\mathrm{fin}}$ and $\underset{\sim}{\mathbf{M}}$ is injective in $\mathcal{X}_{\mathrm{fin}}$.*

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For $k \geqslant 2$, a $(k+1)$-ary term $n(v_1, \dots, v_{k+1})$ is called a near unanimity term or NU term for an algebra $\underline{\mathbf{M}}$ if $\underline{\mathbf{M}}$ satisfies

$$n(y, x, \dots, x) \approx n(x, y, x, \dots, x) \approx \cdots \approx n(x, \dots, x, y) \approx x.$$

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Lemma (2.3.3 NU Lemma)

(*K. Baker and A. Pixley*) *Let $k \geq 2$ and assume that $\underline{\mathbf{M}}$ has a $(k+1)$-ary NU term. Let X be a subset of M^m and let $\alpha\colon X \to M$ be a map that preserves every relation in R_k. Then α preserves every relation in R_ω.*

The NU Duality Theorem

The following useful result is an immediate corollary.

Theorem (NU Duality Theorem)

Assume that $\underline{\mathbf{M}}$ *is a finite algebra that has a* $(k+1)$*-ary NU term. Then* $\underset{\sim}{\mathbf{M}} := \langle M; R_k, \mathcal{T} \rangle$ *yields a duality on* $\mathcal{A}$ *and is injective in* $\mathcal{X}$.

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$$m(x, y, z) := (x \wedge y) \vee (y \wedge z) \vee (z \wedge x).$$

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Thus we obtain the most widely used result in the theory.

Theorem (Lattice-based Duality Theorem)

Let $\mathbf{\underline{M}}$ *be a finite lattice-based algebra. Then* $\underset{\sim}{\mathbf{M}} := \langle M; R_2, \mathcal{T} \rangle$ *yields a duality on* $\mathcal{A}$ *and is injective in* $\mathcal{X}$*.*

Priestley duality via the Lattice-based Duality Theorem

In Lecture 2 we saw how to obtain (half of) Priestley duality from the Second Duality Theorem. As an application of the Lattice-based Duality Theorem, it is almost immediate.

- $\underline{\mathbf{D}} = \langle \{0, 1\}; \vee, \wedge, 0, 1 \rangle$ and $\underset{\sim}{\mathbf{D}} = \langle \{0, 1\}; \leqslant, \mathcal{T} \rangle$.

Theorem (Half of Priestley duality)

$\underset{\sim}{\mathbf{D}}$ yields a duality on the class $\mathcal{D} := \mathrm{ISP}(\underline{\mathbf{D}})$ of bounded distributive lattices, i.e., $e_{\mathbf{A}} : \mathbf{A} \to ED(\mathbf{A})$ is an isomorphism, for all $\mathbf{A} \in \mathcal{D}$.

Priestley duality via the Lattice-based Duality Theorem

We must show that, for all $\mathbf{A} \in \mathfrak{D}$, the evaluation maps

$$e_{\mathbf{A}}(a) \colon \mathfrak{D}(\mathbf{A}, \underline{\mathbf{D}}) \to \{0, 1\},$$

for $a \in A$, are the only continuous order-preserving maps.

Proof.

Let $\alpha \colon \mathfrak{D}(\mathbf{A}, \underline{\mathbf{D}}) \to \{0, 1\}$ be a continuous order-preserving map.

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- By the Lattice-based Duality Theorem, $\underset{\sim}{\mathbf{D}}' := \langle \{0,1\}; R_2, \mathcal{T} \rangle$ yields a duality on $\mathcal{D}$.

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- So the evaluations $e_{\mathbf{A}}(a)$ are the only continuous maps from $\mathcal{D}(\mathbf{A}, \underline{\mathbf{D}})$ to $\{0, 1\}$ that preserve the relations in R_2.

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Refining an alter ego via entailment

Definition (Entainment)

Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$, let $\mathbf{A} \in \mathcal{A}$ and let s be an algebraic relation or (partial) operation on $\underline{\mathbf{M}}$.

- $G \cup H \cup R$ entails s on $D(\mathbf{A})$ if every continuous $G \cup H \cup R$-preserving map $\alpha\colon D(\mathbf{A}) \to M$ preserves s.
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The following lemma is trivial but useful.

Lemma

Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ and $\underset{\sim}{\mathbf{M}}' = \langle M; G', H', R', \mathcal{T}\rangle$ be alter egos of $\underline{\mathbf{M}}$. If $\underset{\sim}{\mathbf{M}}'$ yields a duality of $\mathcal{A}$ and $G \cup H \cup R$ entails s, for all $s \in G' \cup H' \cup R'$, then $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$.

Constructs for entailment

On pages 25–27 of *The Lonely Planet Guide to the Theory of Natural Dualities* there is a list of 15 constructs for entailment. Some are:

(1) **Trivial relations** If θ is an equivalence relation on $\{1, \dots, n\}$ then any $G \cup H \cup R$ entails the relation $\Delta^\theta := \{(c_1, \dots, c_n) \mid i\,\theta\, j \Rightarrow c_i = c_j\}$.
Special cases are Δ_M and M^2.

(4) **Permutation** r entails
$r^\sigma := \{(c_1, \dots, c_n) \mid (c_{\sigma(1)}, \dots, c_{\sigma(n)}) \in r\}$.
Converse $r^{\smile} := \{(c_1, c_2) \mid (c_2, c_1) \in r\}$ is a special case.

(6) **Intersection** If r and s are n-ary, the $\{r, s\}$ entails $r \cap s$.

(7) **Product** $\{r, s\}$ entails $r \times s$.

N.B. A construct that is not allowed is the relational product $r \cdot s$ of two binary relations!

Outline

4.3.9 Natural duality for Kleene algebras

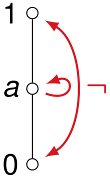
An algebra $\underline{\mathbf{K}} = \langle K; \vee, \wedge, \neg, 0, 1\rangle$ is called a Kleene algebra if it is a bounded distributive lattice satisfying the axioms

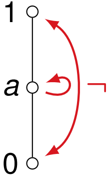
$$\neg(x \wedge y) \approx \neg x \vee \neg y, \quad \neg(x \vee y) \approx \neg x \wedge \neg y, \quad \neg 0 \approx 1,$$

$$\neg\neg x \approx x, \quad x \wedge \neg x \leq y \vee \neg y.$$

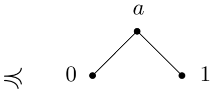
The models of these axioms form a variety $\mathcal{K} = \mathrm{ISP}(\underline{\mathbf{K}})$ generated by the three-element chain

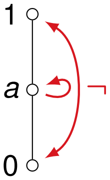
$$\underline{\mathbf{K}} = \langle \{0, a, 1\}; \vee, \wedge, \neg, 0, 1\rangle :$$





- By the Lattice-based Duality Theorem,
  $\underset{\sim}{\mathbf{K}}' := \langle \{0, a, 1\}; R_2, \mathcal{T} \rangle$ yields a duality on $\mathcal{K}$.





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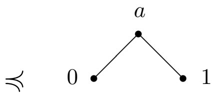
- We must find all subuniverses of $\underline{\mathbf{K}}^2$.



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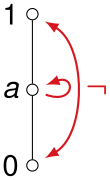

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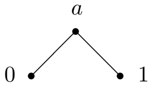
- We must find all subuniverses of $\underline{\mathbf{K}}^2$.
- Let $K_0 = \{0, 1\}$, let $\preccurlyeq = \{00, aa, 11, 0a, 1a\}$ and let $\sim = K^2 \backslash \{01, 10\}$.

$\preccurlyeq$ 

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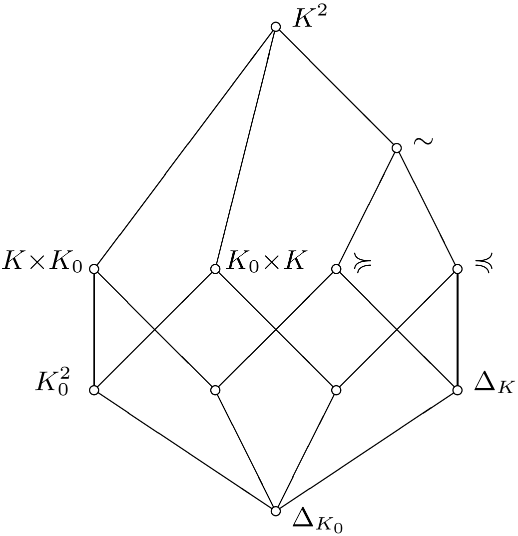

Figure: 8.1 The lattice $\langle R_2; \subseteq \rangle$ of subuniverses of $\underline{\mathbf{K}}^2$

4.3.9 Natural duality for Kleene algebras

Let $R = \{K_0, \preccurlyeq, \sim\}$. Then R entails every relation in R_2 since

- R entails the trivial relation K, whence R entails the products $K \times K_0$, $K_0 \times K$ and $K \times K$,

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- and of course R entails $\sim$ (as $\sim \in R$).

Let $R = \{K_0, \preccurlyeq, \sim\}$. Then R entails every relation in R_2 since

- R entails the trivial relation K, whence R entails the products $K \times K_0$, $K_0 \times K$ and $K \times K$,
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Thus R entails every meet-irreducible relation in the lattice $\langle R_2; \subseteq\rangle$ and so entails every relation in R_2 via intersection.

4.3.9 Natural duality for Kleene algebras

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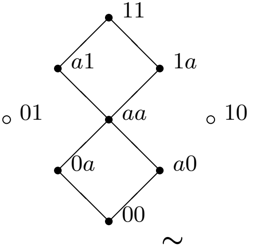
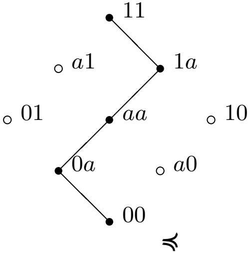
Theorem (Part of 4.3.10)

$\underset{\sim}{\mathbf{K}} = \langle K; K_0, \preccurlyeq, \sim, \mathcal{T}\rangle$ *yields a duality on the class* $\mathcal{K}$ *of Kleene algabras.*





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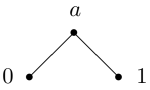

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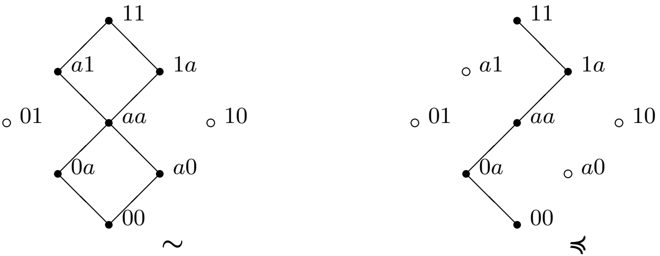

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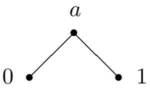

- Note that $\sim \; = \; \succcurlyeq \cdot \preccurlyeq$.

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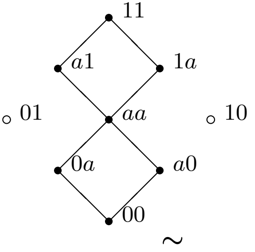

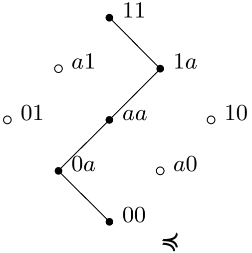

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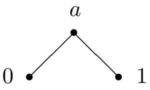

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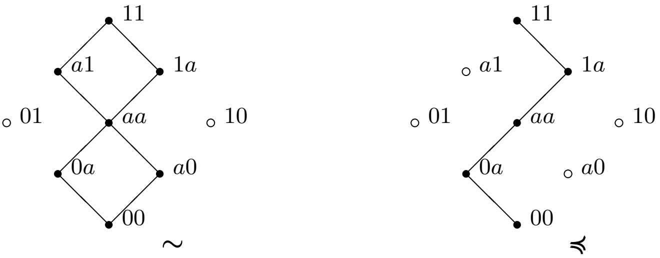
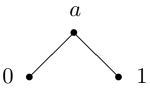


- The uncertainty order on $\{0, a, 1\}$: 

- Note that $\sim \; = \; \succcurlyeq \cdot \preccurlyeq$.
- We will now see that removing $\sim$ will destroy the duality.
- In fact, the duality is optimal.

8.1.3 The Test Algebra Lemma

- Our claim is that, while $\underset{\sim}{\mathbf{K}} = \langle K; K_0, \preccurlyeq, \sim, \mathcal{T} \rangle$ yields a duality on the class $\mathcal{K}$ of Kleene algebras, the alter ego $\underset{\sim}{\mathbf{K}}^* = \langle K; K_0, \preccurlyeq, \mathcal{T} \rangle$ does not.

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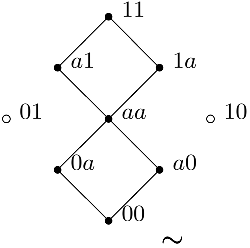
Lemma (Test Algebra Lemma)

Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ and let s be an algebraic relation or (partial) operation on $\underline{\mathbf{M}}$ and let $\mathbf{s}$ be the corresponding subalgebra of $\underline{\mathbf{M}}^n$. Then the following are equivalent:

(i) *$G \cup H \cup R$ entails s;*

(ii) *$G \cup H \cup R$ entails s on $D(\mathbf{s})$.*

Back to the relation $\sim$



$D(\sim) = \mathcal{K}(\sim, \underline{\mathbf{K}}) = \{\rho_1, \rho_2\}$,
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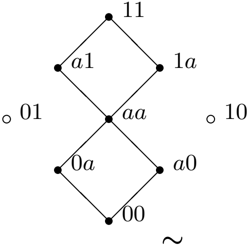

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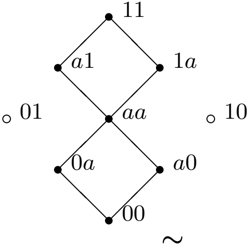

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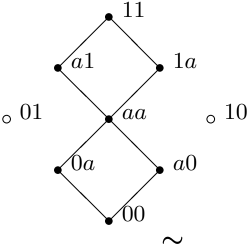

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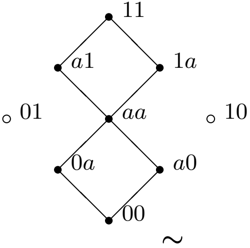

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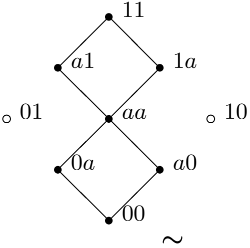

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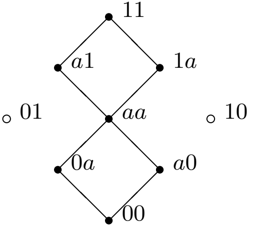

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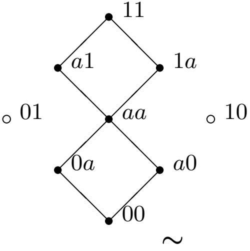

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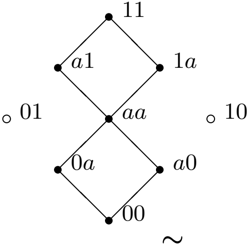

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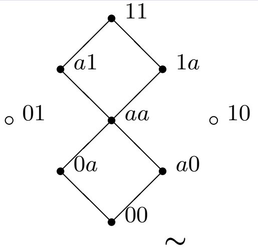

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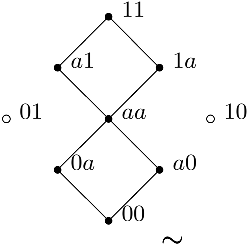

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Outline

Natural dualities: the basics

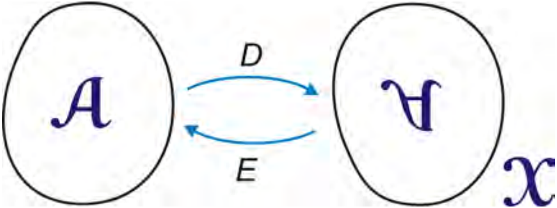
A Natural duality for Kleene algebras

Full and strong dualities

- Full duality
- Strong duality
- The CD Strong Duality Theorem
- Distributive lattices and Kleene algebras revisited
- Partial operations can't be avoided
- Further examples

Full Duality

If $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$ and , in addition, $\varepsilon_{\mathbf{X}}: \mathbf{X} \to DE(\mathbf{X})$ is a surjection and therefore an isomorphism, for all $\mathbf{X}$ in $\mathcal{X}$, then $\underset{\sim}{\mathbf{M}}$ yields a full duality on $\mathcal{A}$ (or $\underset{\sim}{\mathbf{M}}$ fully dualises $\underline{\mathbf{M}}$).



Equivalently, $\underset{\sim}{\mathbf{M}}$ yields a full duality on $\mathcal{A}$ if the dual adjunction $\langle D, E, e, \varepsilon\rangle$ is a dual category equivalence between $\mathcal{A}$ and $\mathcal{X}$.

Strong duality

Let $\underset{\sim}{\mathbf{M}}$ be any alter ego of an algebra $\underline{\mathbf{M}}$, and let

$$D\colon \mathcal{A} \to \mathfrak{X} \qquad \text{and} \qquad E\colon \mathfrak{X} \to \mathcal{A}$$

be the induced hom-functors.

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- $\underset{\sim}{\mathbf{M}}$ is injective in the category $\mathcal{X}$ if, for every embedding $\varphi\colon \mathbf{X} \rightarrowtail \mathbf{Y}$ and every morphism $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ in $\mathcal{X}$, there is a morphism $\beta\colon \mathbf{Y} \to \underset{\sim}{\mathbf{M}}$ such that $\beta \circ \varphi = \alpha$.

$$\begin{tikzcd}
\mathbf{X} \arrow[r, tail, "\varphi"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circ"] & \mathbf{Y} \arrow[dl, dashed, "\beta"] \\
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Strong duality

If $\underset{\sim}{\mathbf{M}}$ fully dualises $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}$ is injective in $\mathfrak{X}$ (so that surjections in $\mathcal{A}$ correspond to embeddings in $\mathfrak{X}$), we say that $\underset{\sim}{\mathbf{M}}$ yields a strong duality on $\mathcal{A}$ (or that $\underset{\sim}{\mathbf{M}}$ strongly dualises $\underline{\mathbf{M}}$).

The CD Strong Duality Theorem

Let $\underline{\mathbf{M}}$ be a finite algebra.

- For all $\mathbf{N} \leqslant \underline{\mathbf{M}}$ define $\mathrm{irr}(\mathbf{N})$ to be the least ℓ such that $0_{\mathbf{N}}$ in $\mathrm{Con}(\mathbf{N})$ is a meet of ℓ meet-irreducible congruences.

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- Define $\mathrm{Irr}(\underline{\mathbf{M}}) := \max\{\, \mathrm{irr}(\mathbf{N}) \mid \mathbf{N} \text{ is a subalgebra of } \underline{\mathbf{M}}\,\}$. $\mathrm{Irr}(\underline{\mathbf{M}})$ is called the irreducibility index of $\underline{\mathbf{M}}$.

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- Define $C := \{a \in M \mid \{a\} \text{ is a subuniverse of } \underline{\mathbf{M}} \,\}$ regarded as a set of nullary operations on M.

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The CD Strong Duality Theorem

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Theorem (3.3.7 CD Strong Duality Theorem)

Assume that $\underline{\mathbf{M}}$ is a finite algebra and that $\underset{\sim}{\mathbf{M}} := \langle M; R, \mathcal{T}\rangle$ dualises $\underline{\mathbf{M}}$. If $\mathrm{Var}(\underline{\mathbf{M}})$ is congruence distributive and $\mathrm{Irr}(\underline{\mathbf{M}}) = n$, then $\underset{\sim}{\mathbf{M}} := \langle M; C \cup H_n, R, \mathcal{T}\rangle$ strongly dualises $\underline{\mathbf{M}}$.

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N.B. $\operatorname{Var}(\underline{\mathbf{M}})$ is congruence distributive if $\underline{\mathbf{M}}$ is lattice based.

Distributive lattices revisited

- $\underline{\mathbf{D}} = \langle \{0,1\}; \vee, \wedge, 0, 1 \rangle$ and $\underset{\sim}{\mathbf{D}} = \langle \{0,1\}; \leqslant, \mathcal{T} \rangle$.

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$\underset{\sim}{\mathbf{D}}$ yields a strong duality between the class $\mathfrak{D} := \mathsf{ISP}(\underline{\mathbf{D}})$ of bounded distributive lattices and the class $\mathcal{P} = \mathsf{IS_cP^+}(\underset{\sim}{\mathbf{D}})$ of Priestley spaces, i.e., $\underset{\sim}{\mathbf{D}}$ is injective in $\mathcal{P}$ and, for all $\mathbf{A} \in \mathfrak{D}$ and $\mathbf{X} \in \mathcal{P}$,

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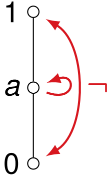
Kleene algebras revisited



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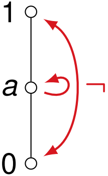

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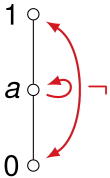

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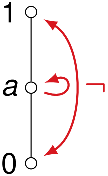

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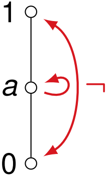

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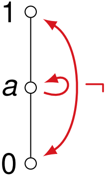

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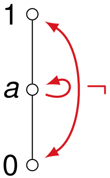

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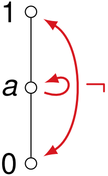

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Partial operations can't be avoided

Theorem (6.1.2 Total Structure Theorem)

Assume that $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T} \rangle$ *yields a strong duality on* $\mathcal{A}$*. The following are equivalent:*

(i) *some total structure* $\underset{\sim}{\mathbf{M}}'$ *yields a strong duality on* $\mathcal{A}$*;*

(ii) *for each natural number* n*, every* n*-ary partial operation* $h \in H$ *extends to a homomorphism* $g \colon \underline{\mathbf{M}}^n \to \underline{\mathbf{M}}$*;*

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- there is an alter ego $\underset{\sim}{\mathbf{M}}$ that yields a strong duality on $\mathcal{A}$,
- but any such $\underset{\sim}{\mathbf{M}}$ must include partial operations in its type.

Further examples

Some exercises for you. Use the Lattice-based Duality Theorem and the CD Strong Duality Theorem to find a strong duality for $\mathcal{A} := \mathbb{ISP}(\underline{\mathbf{M}})$ in each of the following cases.
Is your duality optimal?

1. Median algebras. $\underline{\mathbf{M}} = \langle \{0, 1\}; m \rangle$, where $m\colon \{0, 1\}^3 \to \{0, 1\}$ is the median operation.
2. Stone algebras. $\underline{\mathbf{M}} = \langle \{0, a, 1\}; \vee, \wedge, ^*, 0, 1 \rangle$, where $\langle \{0, a, 1\}; \vee, \wedge, 0, 1 \rangle$ is a chain with $0 < a < 1$ and * is given by $0^* = 1$ and $a^* = 1^* = 0$.
3. Double Stone algebras. $\underline{\mathbf{M}} = \langle \{0, a, b, 1\}; \vee, \wedge, ^*, ^+, 0, 1 \rangle$, where $\langle \{0, a, b, 1\}; \vee, \wedge, 0, 1 \rangle$ is a chain with $0 < a < b < 1$ and * and $^+$ are given by $0^* = 1$ and $a^* = b^* = 1^* = 0$, and $1^+ = 0$ and $0^+ = a^+ = b^+ = 1$.
4. 3-valued Gödel algebras. $\underline{\mathbf{M}} = \langle \{0, a, 1\}; \vee, \wedge, \to, 0, 1 \rangle$, where $\langle \{0, a, 1\}; \vee, \wedge, 0, 1 \rangle$ is a chain with $0 < a < 1$ and $x \to y = 1$, if $x \leqslant y$, and $x \to y = y$, if $x > y$.

Hom-closed and term-closed sets

It is easy to prove the following claims, for all $\mathbf{A} \in \mathcal{A}$.

- The set $D(\mathbf{A}) = \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ is closed under every I-ary algebraic partial operation on $\underline{\mathbf{M}}$, for all non-empty sets I. We say that $D(\mathbf{A})$ is hom-closed in M^A.

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- For all $y \in M^A \backslash \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$, there exist A-ary term functions $t_1, t_2 \colon M^A \to M$ such that $t_1{\restriction}_{D(\mathbf{A})} = t_2{\restriction}_{D(\mathbf{A})}$ but $t_1(y) \neq t_2(y)$.
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Hom-closed and term-closed sets

It is easy to prove the following claims, for all $\mathbf{A} \in \mathcal{A}$.

- The set $D(\mathbf{A}) = \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ is closed under every I-ary algebraic partial operation on $\underline{\mathbf{M}}$, for all non-empty sets I. We say that $D(\mathbf{A})$ is hom-closed in M^A.
- For all $y \in M^A \backslash \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$, there exist A-ary term functions $t_1, t_2 \colon M^A \to M$ such that $t_1{\restriction}_{D(\mathbf{A})} = t_2{\restriction}_{D(\mathbf{A})}$ but $t_1(y) \neq t_2(y)$. We say that $D(\mathbf{A})$ is term-closed in M^A.

Theorem (3.2.4 First Strong Duality Theorem)

Assume $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$. The following are equivalent:

(1) $\underset{\sim}{\mathbf{M}}$ *yields a strong duality on* $\mathcal{A}$,

(2) *for every non-empty set S, each closed substructure of $\underset{\sim}{\mathbf{M}}^S$ is hom-closed,*

(3) *for every non-empty set S, each closed substructure of $\underset{\sim}{\mathbf{M}}^S$ is term-closed.*