Lecture 2: An invitation to natural dualities

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Outline

Examples of natural dualities

Natural dualities: the basics

Duality theorems

Boolean algebras — Stone duality

\[ \mathcal{B} = \text{ISP}(\mathcal{B}), \quad \mathcal{B} = \langle \{0, 1\}; \lor, \land, ', 0, 1 \rangle \]
\[ D(A) := \mathcal{B}(A, \mathcal{B}) \leq \mathcal{B}^A \]

Boolean spaces

(i.e., compact, Hausdorff and a basis of clopen sets)

\[ \mathcal{Z} = \text{ISP}'(\mathcal{B}), \quad \mathcal{Z} = \langle \{0, 1\}; \lor, \land, 0, 1 \rangle \]
\[ 0 \quad 1 \]

\[ E(X) := \mathcal{Z}(X, \mathcal{B}) \leq \mathcal{B}^X \]

Boundedly distributive lattices — Priestley duality

\[ \mathcal{D} = \text{ISP}(\mathcal{D}), \quad \mathcal{D} = \langle \{0, 1\}; \lor, \land, 0, 1 \rangle \]
\[ D(A) := \mathcal{D}(A, \mathcal{D}) \leq \mathcal{D}^A \]

\[ \mathcal{P} = \text{ISP}'(\mathcal{D}), \quad \mathcal{P} = \langle \{0, 1\}; \leq \rangle \]
\[ E(X) := \mathcal{P}(X, \mathcal{D}) \leq \mathcal{D}^X \]
Abelian groups — Pontryagin duality

\[ A = \text{ISP}(T), \text{ where } \]

\[ T = \langle T; 1,1 \rangle \]

and \( T := \{ z \in \mathbb{C} : |x| = 1 \} \)

\[ D(A) := A(A, T) \leq T^A \]

Generalising to natural dualities: “why bother?”

Let \( M = \langle M; F \rangle \) be one of \( B, D \) and \( T \), and let \( M = \langle M; G, R, T \rangle \) be the corresponding topological structure, \( B, D \) or \( T \).

- A duality for \( A := \text{ISP}(M) \) gives a uniform way to represent each algebra \( A \) as an algebra of continuous functions.
- If we have a full duality and have axiomatised the class \( \mathcal{X} := \text{IScP}(M) \), we can find examples of algebras in \( A \) by simply constructing objects in \( \mathcal{X} \).
- Some dualities have the powerful property of being “logarithmic”—they turn products into sums; e.g., in both \( B \) and \( D \) we have \( D(A \times B) \cong D(A) \cup D(B) \).

Some observations on \( B, D \) and \( A \)

For the functors \( D \) and \( E \) to be well defined, we need the algebras \( B \), \( D \) and \( T \) and the corresponding topological structures \( B, D \) and \( T \) to be compatible.

Let \( M = \langle M; F \rangle \) be one of \( B, D \) and \( T \), and let \( M = \langle M; G, R, T \rangle \) be the corresponding topological structure, \( B, D \) or \( T \).

Define \( A := \text{ISP}(M) \) and \( \mathcal{X} := \text{IScP}(M) \), and let \( A \in A \) and \( X \in \mathcal{X} \).

Since we define \( D(A) := A(A, M) \) and \( E(X) := X(X, M) \), in order to have \( D(A) \in \text{IScP}(M) \) and \( E(X) \in \text{ISP}(M) \), we need

- \( A(A, M) \) to be a topologically closed substructure of \( M^A \), and
- \( X(X, M) \) to be a subalgebra of \( M^X \).
Some observations

Let $\mathbf{M} = (M; F)$, let $\mathbf{M} = (M; G, R, \mathcal{T})$, define $\mathcal{A} := \text{ISP}(\mathbf{M})$ and $\mathcal{X} := \text{ISP}^p(\mathbf{M})$, and let $A \in \mathcal{A}$ and $X \in \mathcal{X}$. We need $\mathbf{M}$ and $\mathbf{M}$ to be compatible in such a way that

- $\mathcal{A}(A, M)$ is a topologically closed substructure of $M^A$, and
- $\mathcal{X}(X, M)$ is a subalgebra of $M^X$.

- $\mathcal{A}(A, M)$ will be topologically closed in $M^A$, provided the topology on $M$ is Hausdorff and the operations in $F$ are continuous. (If $M$ is compact, then so is $\mathcal{A}(A, M)$.)
- $\mathcal{A}(A, M)$ will be closed under the operations in $G$ provided each $(n$-ary) $g \in G$ is a homomorphism from $M^n$ to $M$.
- $\mathcal{X}(X, M)$ will be a subalgebra of $M^X$ provided
  - each $(n$-ary) $g \in G$ is a homomorphism from $M^n$ to $M$,
  - each $(n$-ary) relation $r \in R$ is a subuniverse of $M^n$, and
  - each operation in $F$ is continuous.

When these highlighted conditions hold, we say that $g$ and $r$ are compatible with or algebraic over $M$.

Natural dualities: categories and functors

Let $\mathbf{M} = (M; G, H, R, \mathcal{T})$ be an alter ego of $\mathbf{M}$.

The categories $\mathcal{A}$ and $\mathcal{X}$

- Define $\mathcal{A} := \text{ISP}(\mathbf{M})$: the algebraic category of interest.
- Define $\mathcal{X} := \text{ISP}^p(\mathbf{M})$: the potential dual category for $\mathcal{A}$.

The contravariant functors $D$ and $E$

- There are natural hom-functors $D: \mathcal{A} \to \mathcal{X}$ and $E: \mathcal{X} \to \mathcal{A}$.
- For each algebra $A$ in $\mathcal{A}$, the underlying set of $D(A)$ is the set $\text{hom}(A, M)$ of all homomorphisms from $A$ into $M$, and $D(A)$ is a topologically closed substructure of $M^A$.
- For each structure $X$ in $\mathcal{X}$, the underlying set of $E(X)$ is the set $\text{hom}(X, M)$ of all continuous homomorphisms from $X$ into $M$, and $E(X)$ is a subalgebra of $M^X$.

Natural dualities: alter egos

Generalizing our examples, we start with an algebra $\mathbf{M}$ and wish to find a dual category for the prevariety $\mathcal{A} := \text{ISP}(\mathbf{M})$.

An alter ego of an algebra

A structure $\mathbf{M} = (M; G, H, R, \mathcal{T})$ is an alter ego of $\mathbf{M}$ if it is compatible with $\mathbf{M}$, that is,

- $G$ is a set of operations on $M$, each of which is a homomorphism with respect to $\mathbf{M}$,
- $H$ is a set of partial operations on $M$, each of which is a homomorphism with respect to $\mathbf{M}$,
- $R$ is a set of relations on $M$, each of which is a subuniverse of the appropriate power of $\mathbf{M}$, and
- $\mathcal{T}$ is a compact Hausdorff topology on $M$ with respect to which the operations on $\mathbf{M}$ are continuous.

Natural dualities: embeddings

Natural embeddings

For all $A \in \mathcal{A}$ and $X \in \mathcal{X}$, there are embeddings

$e_A : A \to ED(A) = \mathcal{X}(\mathcal{A}(A, M), M)$, given by
$(\forall a \in A) e_A(a) : \mathcal{A}(A, M) \to M$ with
$(\forall x \in \mathcal{A}(A, M)) e_A(a)(x) := x(a)$,

and
$e_X : X \to DE(X) = \mathcal{A}(\mathcal{X}(X, M), M)$, given by
$(\forall x \in X) e_X(x) : \mathcal{X}(X, M) \to M$ with
$(\forall \alpha \in \mathcal{X}(X, M)) e_X(x)(\alpha) := \alpha(x)$.

These embeddings yield natural transformations

e : \text{id}_\mathcal{A} \to ED and \ \eps : \text{id}_\mathcal{X} \to DE,

and $\langle D, E, e, \eps \rangle$ is a dual adjunction between $\mathcal{A}$ and $\mathcal{X}$.
A dual adjunction

\[
\begin{array}{ccc}
A & \xrightarrow{u} & B \\
\varepsilon_A & & \varepsilon_B \\
ED(A) & \xrightarrow{ED(u)} & ED(B)
\end{array}
\quad \begin{array}{ccc}
X & \xrightarrow{\varphi} & Y \\
\varepsilon_X & & \varepsilon_Y \\
DE(X) & \xrightarrow{DE(\varphi)} & DE(Y)
\end{array}
\quad \begin{array}{ccc}
A & \xrightarrow{\varepsilon_A} & ED(A) \\
& & \varepsilon_X \\
& & DE(X)
\end{array}
\quad \begin{array}{ccc}
X & \xrightarrow{\varphi} & DE(X) \\
& & D(u) \\
& & D(A)
\end{array}
\]

- For \( u : A \to B \) and \( \varphi : X \to Y \), the two squares commute.
- \( \mathcal{A}(A, E(X)) \cong \mathcal{X}(X, D(A)) \) via the triangles:
  \[ u = E(D(u) \circ \varepsilon_X) \circ \varepsilon_A \text{ and } \varphi = D(E(\varphi) \circ \varepsilon_A) \circ \varepsilon_X. \]

Full duality

If, in addition, \( \varepsilon_X : X \to DE(X) \) is a surjection and therefore an isomorphism, for all \( X \) in \( \mathcal{X} \), then \( \mathcal{M} \) yields a full duality on \( \mathcal{A} \) (or \( \mathcal{M} \) fully dualises \( \mathcal{M} \)).

Duality

If \( \varepsilon_A : A \to ED(A) \) is surjective and therefore an isomorphism, for all \( A \) in \( \mathcal{A} \), then we say that \( \mathcal{M} \) yields a duality on \( \mathcal{A} \) (or that \( \mathcal{M} \) dualises \( \mathcal{M} \)).

\[ \begin{array}{ccc}
& \mathcal{A} & \\
D & \searrow & E \\
\mathcal{N} & \nearrow & \mathcal{X}
\end{array} \]

Equivalently, \( \mathcal{M} \) yields a duality on \( \mathcal{A} \) if the dual adjunction \( \langle D, E, e, \varepsilon \rangle \) is a dual category equivalence between \( \mathcal{A} \) and \( \mathcal{X} \).

Embeddings, injectivity and strong duality

Let \( \mathcal{M} \) be any alter ego of an algebra \( \mathcal{M} \), and let

\[ D : \mathcal{A} \to \mathcal{X} \quad \text{and} \quad E : \mathcal{X} \to \mathcal{A} \]

be the induced hom-functors.

It is easy to see that:

- \( D \) and \( E \) send surjections to embeddings,
- \( D \) sends embeddings in \( \mathcal{A} \) to surjections in \( \mathcal{X} \) if and only if \( \mathcal{M} \) is injective in \( \mathcal{A} \), and
- \( E \) sends embeddings in \( \mathcal{X} \) to surjections in \( \mathcal{A} \) if and only if \( \mathcal{M} \) is injective in \( \mathcal{X} \).

Strong duality

If \( \mathcal{M} \) fully dualises \( \mathcal{M} \) and \( \mathcal{M} \) is injective in \( \mathcal{X} \) (so that surjections in \( \mathcal{A} \) correspond to embeddings in \( \mathcal{X} \)), we say that \( \mathcal{M} \) yields a strong duality on \( \mathcal{A} \) (or that \( \mathcal{M} \) strongly dualises \( \mathcal{M} \)).
Further examples

- All three of our original examples — Stone duality, Priestley duality and Pontryagin duality — are examples of strong dualities.
- Every finite lattice-based algebra admits a strong duality. [Davey, Werner 1980 and Clark, Davey 1995]
- The unary algebra $\mathcal{I}$ admits a duality, but not a full duality. [Hyndman, Willard 2000]
- There is an example of a three-element algebra that admits a full duality that can not be upgraded to a strong duality. [Pitkethly 2009]
- The two-element implication algebra $I := \langle \{0, 1\}; \rightarrow \rangle$ does not admit a natural duality. [Davey, Werner 1980]

Duals of free algebras

- Given a non-empty set $S$, the set $\mathbb{F}_M(S) = \{ t : M^S \rightarrow M \mid t \text{ is an } S\text{-ary term function on } M \}$ is the free $S$-generated algebra in $\mathcal{A}$ (the projections $\pi_s : M^S \rightarrow M$, for $s \in S$, are the free generators).

Lemma (2.2.1)

Let $S$ be a non-empty set. The then dual of $\mathbb{F}_M(S)$, namely $D(\mathbb{F}_M(S)) = \mathcal{A}(\mathbb{F}_M(S), M)$, is isomorphic in $\mathcal{X}$ to $M^S$.

- It is easy to see that every $S$-ary term function $t$ on $M$ is an $\mathcal{X}$-morphism, i.e., $t : M^S \rightarrow M$.
- If $M$ yields a duality on $\mathcal{A}$, then $\mathbb{F}_M(S) \cong E(D(\mathbb{F}_M(S))) \cong E(M^S) \cong \mathcal{X}(M^S, M)$.

The interpolation condition (IC)

Let $\mathcal{A}_{\text{fin}}$ and $\mathcal{X}_{\text{fin}}$ consist of the finite members of $\mathcal{A}$ and $\mathcal{X}$.

Lemma (2.2.5)

The following are equivalent:

(i) (IC) for each $n \in \mathbb{N}$ and each substructure $X$ of $M^n$, every morphism $\alpha : X \rightarrow M$ extends to a term function $t : M^n \rightarrow M$ of the algebra $M$.

(ii) (INJ)$^{\mathcal{A}_{\text{fin}}}$ $M$ is injective in $\mathcal{X}_{\text{fin}}$, and (CLO) for each $n \in \mathbb{N}$, every morphism $t : M^n \rightarrow M$ is an $n$-ary term function on $M$.

(iii) $M$ yields a duality on $\mathcal{A}_{\text{fin}}$ and is injective in $\mathcal{X}_{\text{fin}}$.

We would like to obtain a duality for $\mathcal{A}$ in two steps:

- first show that $M$ yields a duality on $\mathcal{A}_{\text{fin}}$, then
- apply some general theory to show that the duality lifts automatically to a duality on the whole of $\mathcal{A}$.

This is achievable provided $M$ enjoys some degree of finiteness.

For duality, relations will do

- Let $M = \langle M; F \rangle$ be a finite algebra,
- let $M = \langle M; G, H, R, \mathcal{T} \rangle$ be an alter ego of $M$, and
- define $\mathcal{A} := \text{ISP}(M)$ and $\mathcal{X} := \text{ISP}^+(M)$.

Recall that to prove that $M$ yields a duality on $\mathcal{A}$, it remains to show that

- for all $A \in \mathcal{A}$, the evaluation maps $e_A$, for $a \in A$, are the only $\mathcal{X}$-morphisms from $\mathcal{A}(A, M)$ to $M$.

Lemma (2.1.2)

Let $M = \langle M; G, H, R, \mathcal{T} \rangle$, define $M' = \langle M; R', \mathcal{T} \rangle$ where

$R' := R \cup \{ \text{graph}(h) \mid h \in G \cup H \}$

Then $M$ yields a duality on $\mathcal{A}$ if and only if $M'$ does.

- Thus, as far as obtaining a duality is concerned, we can restrict our attention to purely relational alter egos.
The Second Duality Theorem

If \( M = \langle M; G, R, \mathcal{T} \rangle \), that is, the type of \( M \) includes no partial operations, then we call \( M \) a total structure.

Theorem (2.2.7 Second Duality Theorem)
Assume that \( M \) is a total structure with \( R \) finite. If (IC) holds, then \( M \) yields a duality on \( \mathcal{A} \) and is injective in \( \mathcal{X} \).

This result is rather surprising.

- It gives us simple finitary conditions which yield both a dual adjunction between the categories \( \mathcal{A} \) and \( \mathcal{X} \) and a topological representation of every algebra in \( \mathcal{A} \),
- but it requires us to do no category theory and no topology!

Some exercises for you. In each case, prove that (IC) holds.

1. [Stone] Let \( \mathcal{B} = \langle \{0, 1\}; \lor, \land, 0, 1 \rangle \); then \( \mathcal{B} = \text{ISP(} \mathcal{B} \text{)} \) is the class of Boolean algebras. Show that \( \mathcal{B} = \langle \{0, 1\}; \mathcal{T} \rangle \) yields a duality on \( \mathcal{B} \).
2. [Priestley] Let \( \mathcal{L} = \langle \{0, 1\}; \lor, \land \rangle \); then \( \mathcal{L} = \text{ISP(} \mathcal{L} \text{)} \) is the class of distributive lattices. Show that \( \mathcal{L} = \langle \{0, 1\}; 0, 1, \leq, \mathcal{T} \rangle \) yields a duality on \( \mathcal{L} \).
3. [Hofmann–Elves–Stralka] Let \( \mathcal{S} = \langle \{0, 1\}; \land \rangle \); then \( \mathcal{S} = \text{ISP(} \mathcal{S} \text{)} \) is the class of meet semilattices. Show that \( \mathcal{S} = \langle \{0, 1\}; \land, 0, \mathcal{T} \rangle \) yields a duality on \( \mathcal{S} \).
4. [Pontryagin] Let \( \mathcal{Z}_m = \langle \mathbb{Z}_m; +, -, 0 \rangle \); then \( \mathcal{A}_m = \text{ISP(} \mathcal{Z}_m \text{)} \) is the class of abelian groups of exponent \( m \). Show that \( \mathcal{Z}_m = \langle \mathbb{Z}_m; +, -, 0, \mathcal{T} \rangle \) yields a duality on \( \mathcal{A}_m \).