Lecture 2: An invitation to natural dualities

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Outline

Examples of natural dualities

Natural dualities: the basics

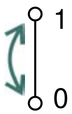
Duality theorems

Boolean algebras — Stone duality

Boolean algebras

$\mathcal{B} = \mathrm{ISP}(\underline{\mathbf{B}})$, where

$\underline{\mathbf{B}} = \langle \{0,1\}; \vee, \wedge, ', 0, 1\rangle$



$D(\mathbf{A}) := \mathcal{B}(\mathbf{A}, \underline{\mathbf{B}}) \leqslant \underset{\sim}{\mathbf{B}}^A$

Boolean spaces

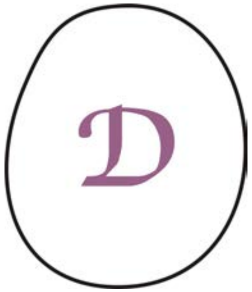
(i.e., compact, Hausdorff and a basis of clopen sets)

$\mathcal{Z} = \mathrm{IS_cP^+}(\underset{\sim}{\mathbf{B}})$, where

$\underset{\sim}{\mathbf{B}} = \langle \{0,1\}; \mathcal{T}\rangle$

$E(\mathbf{X}) := \mathcal{Z}(\mathbf{X}, \underset{\sim}{\mathbf{B}}) \leqslant \mathbf{B}^X$

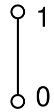
Bounded distributive lattices — Priestley duality



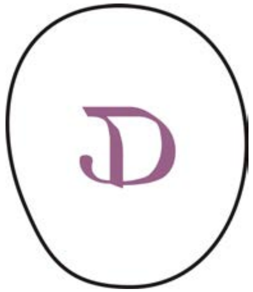
Bounded distributive lattices

$\mathcal{D} = \mathrm{ISP}(\underline{\mathbf{D}})$, where

$\underline{\mathbf{D}} = \langle \{0,1\}; \vee, \wedge, 0, 1\rangle$



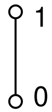
$D(\mathbf{A}) := \mathcal{D}(\mathbf{A}, \underline{\mathbf{D}}) \leqslant \underset{\sim}{\mathbf{D}}^A$



Priestley spaces

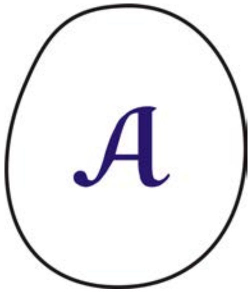
$\mathcal{P} = \mathrm{IS_cP^+}(\underset{\sim}{\mathbf{D}})$, where

$\underset{\sim}{\mathbf{D}} = \langle \{0,1\}; \leqslant, \mathcal{T}\rangle$



$E(\mathbf{X}) := \mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{D}}) \leqslant \underline{\mathbf{D}}^X$

Abelian groups — Pontryagin duality



Abelian groups

$\mathcal{A} = \mathrm{ISP}(\underline{\mathbf{T}})$, where

$\underline{\mathbf{T}} = \langle T; \cdot, {}^{-1}, 1\rangle$

and $T := \{z \in \mathbb{C} : |x| = 1\}$

$D(\mathbf{A}) := \mathcal{A}(\mathbf{A}, \underline{\mathbf{T}}) \leqslant \underset{\sim}{\mathbf{T}}^A$

Compact top. abelian groups

$\mathcal{X} = \mathrm{IS_cP^+}(\underset{\sim}{\mathbf{T}})$, where

$\underset{\sim}{\mathbf{T}} = \langle T; \cdot, {}^{-1}, 1, \mathcal{T}\rangle$

$E(\mathbf{X}) := \mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{T}}) \leqslant \mathbf{T}^X$

Generalising to natural dualities: “why bother?"

Let $\underline{\mathbf{M}} = \langle M; F\rangle$ be one of $\underline{\mathbf{B}}$, $\underline{\mathbf{D}}$ and $\underline{\mathbf{T}}$, and let $\underset{\sim}{\mathbf{M}} = \langle M; G, R, \mathcal{T}\rangle$ be the corresponding topological structure, $\underset{\sim}{\mathbf{B}}$, $\underset{\sim}{\mathbf{D}}$ or $\underset{\sim}{\mathbf{T}}$.

- A duality for $\mathcal{A} := \mathrm{ISP}(\underline{\mathbf{M}})$ gives a uniform way to represent each algebra $\mathbf{A} \in \mathcal{A}$ as an algebra of continuous functions.
- If we have a full duality and have axiomatised the class $\mathcal{X} := \mathrm{IS_cP^+}(\underset{\sim}{\mathbf{M}})$, we can find examples of algebras in $\mathcal{A}$ by simply constructing objects in $\mathcal{X}$.
- Some dualities have the powerful property of being “logarithmic”—they turn products into sums; e.g., in both $\mathcal{B}$ and $\mathcal{D}$ we have $D(\mathbf{A} \times \mathbf{B}) \cong D(\mathbf{A}) \,\dot\cup\, D(\mathbf{B})$.

Generalising to natural dualities: “why bother?"

- Algebraic questions in $\mathcal{A}$ can be answered by translating them into (often simpler) questions in $\mathcal{X}$. For example,
  (1) free algebras in $\mathcal{A}$ are easily described via their duals in $\mathcal{X}$,
  (2) while a coproduct $\mathbf{A} * \mathbf{B}$ is often difficult to describe, its dual, $D(\mathbf{A} * \mathbf{B})$, is simply the cartesian product $D(\mathbf{A}) \times D(\mathbf{B})$,
  (3) congruence lattices in $\mathcal{A}$ may be studied by looking at lattices of closed substructures in $\mathcal{X}$,
  (4) injective algebras in $\mathcal{A}$ may be characterised by first studying projective structures in $\mathcal{X}$,
  (5) algebraically closed and existentially closed algebras may be described via their duals.

Some observations on $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{A}$

For the functors D and E to be well defined, we need the algebras $\underline{\mathbf{B}}$, $\underline{\mathbf{D}}$ and $\underline{\mathbf{T}}$ and the corresponding topological structures $\underset{\sim}{\mathbf{B}}$, $\underset{\sim}{\mathbf{D}}$ and $\underset{\sim}{\mathbf{T}}$ to be compatible.

Let $\underline{\mathbf{M}} = \langle M; F\rangle$ be one of $\underline{\mathbf{B}}$, $\underline{\mathbf{D}}$ and $\underline{\mathbf{T}}$, and let $\underset{\sim}{\mathbf{M}} = \langle M; G, R, \mathcal{T}\rangle$ be the corresponding topological structure, $\underset{\sim}{\mathbf{B}}$, $\underset{\sim}{\mathbf{D}}$ or $\underset{\sim}{\mathbf{T}}$.

Define $\mathcal{A} := \mathrm{ISP}(\underline{\mathbf{M}})$ and $\mathcal{X} := \mathrm{IS_cP^+}(\underset{\sim}{\mathbf{M}})$, and let $\mathbf{A} \in \mathcal{A}$ and $\mathbf{X} \in \mathcal{X}$.

Since we define $D(\mathbf{A}) := \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ and $E(\mathbf{X}) := \mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$, in order to have $D(\mathbf{A}) \in \mathrm{IS_cP^+}(\underset{\sim}{\mathbf{M}})$ and $E(\mathbf{X}) \in \mathrm{ISP}(\underline{\mathbf{M}})$, we need

- $\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ to be a topologically closed substructure of $\underset{\sim}{\mathbf{M}}^A$, and
- $\mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$ to be a subalgebra of $\underline{\mathbf{M}}^X$.

Some observations

Let $\underline{\mathbf{M}} = \langle M; F\rangle$, let $\underset{\sim}{\mathbf{M}} = \langle M; G, R, \mathcal{T}\rangle$, define $\mathcal{A} := \mathsf{ISP}(\underline{\mathbf{M}})$ and $\mathcal{X} := \mathsf{IS_cP^+}(\underset{\sim}{\mathbf{M}})$, and let $\mathbf{A} \in \mathcal{A}$ and $\mathbf{X} \in \mathcal{X}$. We need $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}$ to be compatible in such a way that

- $\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ is a topologically closed substructure of $\underset{\sim}{\mathbf{M}}^A$, and
- $\mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$ is a subalgebra of $\underline{\mathbf{M}}^X$.
  - $\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ will be topologically closed in M^A, provided the topology on M is Hausdorff and the operations in F are continuous. (If $\underset{\sim}{\mathbf{M}}$ is compact, then so is $\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$.)
  - $\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ will be closed under the operations in G provided each (n-ary) $g \in G$ is a homomorphism from $\underline{\mathbf{M}}^n$ to $\underline{\mathbf{M}}$.
  - $\mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$ will be a subalgebra of $\underline{\mathbf{M}}^X$ provided
    - each (n-ary) $g \in G$ is a homomorphism from $\underline{\mathbf{M}}^n$ to $\underline{\mathbf{M}}$,
    - each (n-ary) relation $r \in R$ is a subuniverse of $\underline{\mathbf{M}}^n$, and
    - each operation in F is continuous.

When these highlighted conditions hold, we say that g and r are compatible with or algebraic over $\underline{\mathbf{M}}$.

Natural dualities: alter egos

Generalizing our examples, we start with an algebra $\underline{\mathbf{M}}$ and wish to find a dual category for the prevariety $\mathcal{A} := \mathsf{ISP}(\underline{\mathbf{M}})$.

An alter ego of an algebra

A structure $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ is an alter ego of $\underline{\mathbf{M}}$ if it is compatible with $\underline{\mathbf{M}}$, that is,

- G is a set of operations on M, each of which is a homomorphism with respect to $\underline{\mathbf{M}}$,
- H is a set of partial operations on M, each of which is a homomorphism with respect to $\underline{\mathbf{M}}$,
- R is a set of relations on M, each of which is a subuniverse of the appropriate power of $\underline{\mathbf{M}}$, and
- $\mathcal{T}$ is a compact Hausdorff topology on M with respect to which the operations on $\underline{\mathbf{M}}$ are continuous.

Natural dualities: categories and functors

Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ be an alter ego of $\underline{\mathbf{M}}$.

The categories $\mathcal{A}$ and $\mathcal{X}$

- Define $\mathcal{A} := \mathsf{ISP}(\underline{\mathbf{M}})$: the algebraic category of interest.
- Define $\mathcal{X} := \mathsf{IS_cP^+}(\underset{\sim}{\mathbf{M}})$: the potential dual category for $\mathcal{A}$.

The contravariant functors D and E

- There are natural hom-functors $D\colon \mathcal{A} \to \mathcal{X}$ and $E\colon \mathcal{X} \to \mathcal{A}$.
- For each algebra $\mathbf{A}$ in $\mathcal{A}$, the underlying set of $D(\mathbf{A})$ is the set $\hom(\mathbf{A}, \underline{\mathbf{M}})$ of all homomorphisms from $\mathbf{A}$ into $\underline{\mathbf{M}}$, and $D(\mathbf{A})$ is a topologically closed substructure of $\underset{\sim}{\mathbf{M}}^A$.
- For each structure $\mathbf{X}$ in $\mathcal{X}$, the underlying set of $E(\mathbf{X})$ is the set $\hom(\mathbf{X}, \underset{\sim}{\mathbf{M}})$ of all continuous homomorphisms from $\mathbf{X}$ into $\underset{\sim}{\mathbf{M}}$, and $E(\mathbf{X})$ is a subalgebra of $\underline{\mathbf{M}}^X$.

Natural dualities: embeddings

Natural embeddings

For all $\mathbf{A} \in \mathcal{A}$ and $\mathbf{X} \in \mathcal{X}$, there are embeddings

$$
\begin{aligned}
e_{\mathbf{A}}\colon \mathbf{A} \to ED(\mathbf{A}) = \mathcal{X}(\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}}), \underset{\sim}{\mathbf{M}}), &\quad \text{given by} \\
(\forall a \in A)\ e_{\mathbf{A}}(a)\colon \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}}) \to \underset{\sim}{\mathbf{M}} &\quad \text{with} \\
(\forall x \in \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}}))\ \ e_{\mathbf{A}}(a)(x) := x(a),&
\end{aligned}
$$

and

$$
\begin{aligned}
\varepsilon_{\mathbf{X}}\colon \mathbf{X} \to DE(\mathbf{X}) = \mathcal{A}(\mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}}), \underline{\mathbf{M}}), &\quad \text{given by} \\
(\forall x \in X)\ \varepsilon_{\mathbf{X}}(x)\colon \mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}}) \to \underline{\mathbf{M}} &\quad \text{with} \\
(\forall \alpha \in \mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}}))\ \varepsilon_{\mathbf{X}}(x)(\alpha) := \alpha(x).&
\end{aligned}
$$

These embeddings yield natural transformations

$$
e\colon \mathrm{id}_{\mathcal{A}} \to ED \qquad \text{and} \qquad \varepsilon\colon \mathrm{id}_{\mathcal{X}} \to DE,
$$

and $\langle D, E, e, \varepsilon\rangle$ is a dual adjunction between $\mathcal{A}$ and $\mathcal{X}$.

A dual adjunction

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{u} & \mathbf{B} \\ \downarrow{\scriptstyle e_{\mathbf{A}}} & & \downarrow{\scriptstyle e_{\mathbf{B}}} \\ ED(\mathbf{A}) & \xrightarrow[ED(u)]{} & ED(\mathbf{B}) \end{array} \qquad \begin{array}{ccc} \mathbf{X} & \xrightarrow{\varphi} & \mathbf{Y} \\ \downarrow{\scriptstyle \varepsilon_{\mathbf{X}}} & & \downarrow{\scriptstyle \varepsilon_{\mathbf{Y}}} \\ DE(\mathbf{X}) & \xrightarrow[DE(\varphi)]{} & DE(\mathbf{Y}) \end{array}$$

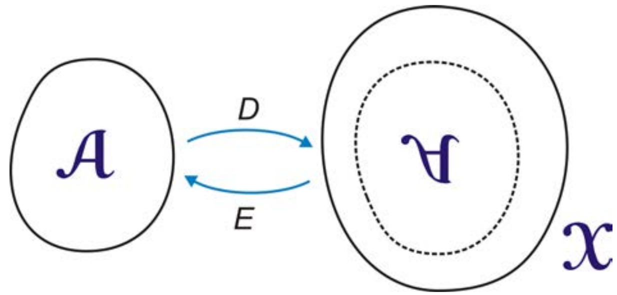
$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{e_{\mathbf{A}}} & ED(\mathbf{A}) \\ & {\scriptstyle u}\searrow & \downarrow{\scriptstyle E(\varphi)} \\ & & E(\mathbf{X}) \end{array} \qquad \begin{array}{ccc} \mathbf{X} & \xrightarrow{\varepsilon_{\mathbf{X}}} & DE(\mathbf{X}) \\ & {\scriptstyle \varphi}\searrow & \downarrow{\scriptstyle D(u)} \\ & & D(\mathbf{A}) \end{array}$$

- For $u\colon \mathbf{A} \to \mathbf{B}$ and $\varphi\colon \mathbf{X} \to \mathbf{Y}$, the two squares commute.
- $\mathcal{A}(\mathbf{A}, E(\mathbf{X})) \cong \mathcal{X}(\mathbf{X}, D(\mathbf{A}))$ via the triangles:

$$u = E(D(u) \circ \varepsilon_{\mathbf{X}}) \circ e_{\mathbf{A}} \text{ and } \varphi = D(E(\varphi) \circ e_{\mathbf{A}}) \circ \varepsilon_{\mathbf{X}}.$$

Duality

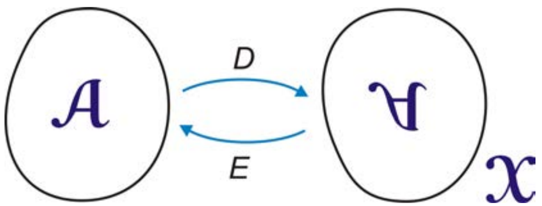
If $e_{\mathbf{A}}\colon \mathbf{A} \to ED(\mathbf{A})$ is surjective and therefore an isomorphism, for all $\mathbf{A}$ in $\mathcal{A}$, then we say that $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$ (or that $\underset{\sim}{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$).



Equivalently, $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$ if the dual adjunction $\langle D, E, e, \varepsilon\rangle$ is a dual category equivalence between $\mathcal{A}$ and a full subcategory of $\mathcal{X}$.

Full duality

If, in addition, $\varepsilon_{\mathbf{X}}\colon \mathbf{X} \to DE(\mathbf{X})$ is a surjection and therefore an isomorphism, for all $\mathbf{X}$ in $\mathcal{X}$, then $\underset{\sim}{\mathbf{M}}$ yields a full duality on $\mathcal{A}$ (or $\underset{\sim}{\mathbf{M}}$ fully dualises $\underline{\mathbf{M}}$).



Equivalently, $\underset{\sim}{\mathbf{M}}$ yields a full duality on $\mathcal{A}$ if the dual adjunction $\langle D, E, e, \varepsilon\rangle$ is a dual category equivalence between $\mathcal{A}$ and $\mathcal{X}$.

Embeddings, injectivity and strong duality

Let $\underset{\sim}{\mathbf{M}}$ be any alter ego of an algebra $\underline{\mathbf{M}}$, and let

$$D\colon \mathcal{A} \to \mathcal{X} \quad \text{and} \quad E\colon \mathcal{X} \to \mathcal{A}$$

be the induced hom-functors.

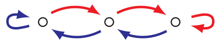
It is easy to see that:

- D and E send surjections to embeddings,
- D sends embeddings in $\mathcal{A}$ to surjections in $\mathcal{X}$ if and only if $\underline{\mathbf{M}}$ is injective in $\mathcal{A}$, and
- E sends embeddings in $\mathcal{X}$ to surjections in $\mathcal{A}$ if and only if $\underset{\sim}{\mathbf{M}}$ is injective in $\mathcal{X}$.

Strong duality

If $\underset{\sim}{\mathbf{M}}$ fully dualises $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}$ is injective in $\mathcal{X}$ (so that surjections in $\mathcal{A}$ correspond to embeddings in $\mathcal{X}$), we say that $\underset{\sim}{\mathbf{M}}$ yields a strong duality on $\mathcal{A}$ (or that $\underset{\sim}{\mathbf{M}}$ strongly dualises $\underline{\mathbf{M}}$).

Further examples

- All three of our original examples — Stone duality, Priestley duality and Pontryagin duality — are examples of strong dualities.
- Every finite lattice-based algebra admits a strong duality. [Davey, Werner 1980 and Clark, Davey 1995]
- The unary algebra  admits a duality, but not a full duality. [Hyndman, Willard 2000]
- There is an example of a three-element algebra that admits a full duality that can not be upgraded to a strong duality. [Pitkethly 2009]
- The two-element implication algebra $\mathbf{I} := \langle \{0,1\}; \to \rangle$ does not admit a natural duality. [Davey, Werner 1980]

For duality, relations will do

- Let $\underline{\mathbf{M}} = \langle M; F \rangle$ be a finite algebra,
- let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T} \rangle$ be an alter ego of $\underline{\mathbf{M}}$, and
- define $\mathcal{A} := \mathsf{ISP}(\underline{\mathbf{M}})$ and $\mathcal{X} := \mathsf{IS_cP^+}(\underset{\sim}{\mathbf{M}})$.

Recall that to prove that $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$, it remains to show that

- for all $\mathbf{A} \in \mathcal{A}$, the evaluation maps $e_{\mathbf{A}}$, for $a \in A$, are the only $\mathcal{X}$-morphisms from $\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ to $\underset{\sim}{\mathbf{M}}$.

Lemma (2.1.2)

Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T} \rangle$, define $\underset{\sim}{\mathbf{M}}' = \langle M; R', \mathcal{T} \rangle$ where

$$R' := R \cup \{\mathrm{graph}(h) \mid h \in G \cup H\}$$

Then $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$ if and only if $\underset{\sim}{\mathbf{M}}'$ does.

- Thus, as far as obtaining a duality is concerned, we can restrict our attention to purely relational alter egos.

Duals of free algebras

- Given a non-empty set S, the set

  $$\mathbf{F}_{\underline{\mathbf{M}}}(S) = \{t \colon M^S \to M \mid t \text{ is an } S\text{-ary term function on } \underline{\mathbf{M}}\}$$

  is the free S-generated algebra in $\mathcal{A}$ (the projections $\pi_s \colon M^S \to M$, for $s \in S$, are the free generators).

Lemma (2.2.1)

Let S be a non-empty set. The then dual of $\mathbf{F}_{\underline{\mathbf{M}}}(S)$, namely

$$D(\mathbf{F}_{\underline{\mathbf{M}}}(S)) = \mathcal{A}(\mathbf{F}_{\underline{\mathbf{M}}}(S), \underline{\mathbf{M}}),$$

is isomorphic in $\mathcal{X}$ to $\underset{\sim}{\mathbf{M}}^S$.

- It is easy to see that every S-ary term function t on $\underline{\mathbf{M}}$ is an $\mathcal{X}$-morphism, i.e., $t \colon \underset{\sim}{\mathbf{M}}^S \to \underset{\sim}{\mathbf{M}}$.
- If $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$, then

  $$\mathbf{F}_{\underline{\mathbf{M}}}(S) \cong ED(\mathbf{F}_{\underline{\mathbf{M}}}(S)) \cong E(\underset{\sim}{\mathbf{M}}^S) \cong \mathcal{X}(\underset{\sim}{\mathbf{M}}^S, \underset{\sim}{\mathbf{M}}).$$

  In fact, we have $\mathbf{F}_{\underline{\mathbf{M}}}(S) = \mathcal{X}(\underset{\sim}{\mathbf{M}}^S, \underset{\sim}{\mathbf{M}})$.

The interpolation condition (IC)

Let $\mathcal{A}_{\mathrm{fin}}$ and $\mathcal{X}_{\mathrm{fin}}$ consist of the finite members of $\mathcal{A}$ and $\mathcal{X}$.

Lemma (2.2.5)

The following are equivalent:

(i) (IC) *for each $n \in \mathbb{N}$ and each substructure $\mathbf{X}$ of $\underset{\sim}{\mathbf{M}}^n$, every morphism $\alpha \colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ extends to a term function $t \colon M^n \to M$ of the algebra $\underline{\mathbf{M}}$,*

(ii) $(\mathrm{INJ})^+_{\mathrm{fin}}$ *$\underset{\sim}{\mathbf{M}}$ is injective in $\mathcal{X}_{\mathrm{fin}}$, and*
(CLO) *for each $n \in \mathbb{N}$, every morphism $t \colon \underset{\sim}{\mathbf{M}}^n \to \underset{\sim}{\mathbf{M}}$ is an n-ary term function on $\underline{\mathbf{M}}$,*

(iii) *$\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}_{\mathrm{fin}}$ and is injective in $\mathcal{X}_{\mathrm{fin}}$.*

We would like to obtain a duality for $\mathcal{A}$ in two steps:

- first show that $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}_{\mathrm{fin}}$, then
- apply some general theory to show that the duality lifts automatically to a duality on the whole of $\mathcal{A}$.

This is achievable provided $\underset{\sim}{\mathbf{M}}$ enjoys some degree of finiteness.

The Second Duality Theorem

If $\underset{\sim}{\mathbf{M}} = \langle M; G, R, \mathcal{T}\rangle$, that is, the type of $\underset{\sim}{\mathbf{M}}$ includes no partial operations, then we call $\underset{\sim}{\mathbf{M}}$ a total structure.

Theorem (2.2.7 Second Duality Theorem)

Assume that $\underset{\sim}{\mathbf{M}}$ is a total structure with R finite. If (IC) holds, then $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$ and is injective in $\mathcal{X}$.

This result is rather surprising.

- It gives us simple finitary conditions which yield both a dual adjunction between the categories $\mathcal{A}$ and $\mathcal{X}$ and a topological representation of every algebra in $\mathcal{A}$,
- but it requires us to do no category theory and no topology!

Priestley duality via the Second Duality Theorem

Recall that

- $\underline{\mathbf{D}} = \langle \{0,1\}; \vee, \wedge, 0, 1\rangle$ is the two-element bounded lattice,
- $\underset{\sim}{\mathbf{D}} = \langle \{0,1\}; \leqslant, \mathcal{T}\rangle$ is the two-element chain endowed with the discrete topology.

Theorem (Half of Priestley duality)

$\underset{\sim}{\mathbf{D}}$ yields a duality on the class $\mathcal{D} := \mathrm{ISP}(\underline{\mathbf{D}})$ of bounded distributive lattices, i.e., $e_{\mathbf{A}} \colon \mathbf{A} \to ED(\mathbf{A})$ is an isomorphism, for all $\mathbf{A} \in \mathcal{D}$.

Proof.

We will prove that (IC) holds. Let $\mathbf{X}$ be a substructure of $\underset{\sim}{\mathbf{D}}^n$ and let $\varphi \colon \mathbf{X} \to \underset{\sim}{\mathbf{D}}$ be a morphism, i.e., φ is order-preserving.

[We need to find a term function $t \colon \{0,1\}^n \to \{0,1\}$ on $\underline{\mathbf{D}}$ such that $t(x) = \varphi(x)$, for all $x \in X$.]

Priestley duality via the Second Duality Theorem

The proof continued

[$\mathbf{X}$ *is a substructure of* $\underset{\sim}{\mathbf{D}}^n$ *and* $\varphi \colon \mathbf{X} \to \underset{\sim}{\mathbf{D}}$ *is order-preserving.*

We need to find a term function $t \colon \{0,1\}^n \to \{0,1\}$ on $\underline{\mathbf{D}}$ such that $t(x) = \varphi(x)$, for all $x \in X$.]

If $\varphi^{-1}(1) = \varnothing$, then define $t(v_1, \dots, v_n) = 0$, and
if $\varphi^{-1}(1) = X$, then define $t(v_1, \dots, v_n) = 1$.

Otherwise, define $t(v_1, \dots, v_n)$ by

$$t(v_1, \dots, v_n) := \bigvee_{a \in \varphi^{-1}(1)} \Big(\bigwedge_{a_i = 1} v_i \Big).$$

Let $x \in X$. If $\varphi(x) = 1$, then $t(x) = 1$, by construction.
If $t(x) = 1$, then there exists $a \in \varphi^{-1}(1)$ with $a_i = 1 \Rightarrow x_i = 1$.
Hence $\varphi(a) = 1$ and $a \leqslant x$. As φ is order-preserving, we have $\varphi(x) = 1$. Hence $t(x) = \varphi(x)$, for all $x \in X$. □

Further applications of the Second Duality Theorem

Some exercises for you. In each case, prove that (IC) holds.

(1) [Stone] Let $\underline{\mathbf{B}} = \langle \{0,1\}; \vee, \wedge, ', 0, 1\rangle$; then $\mathcal{B} = \mathrm{ISP}(\underline{\mathbf{B}})$ is the class of Boolean algebras. Show that $\underset{\sim}{\mathbf{B}} = \langle \{0,1\}; \mathcal{T}\rangle$ yields a duality on $\mathcal{B}$.

(2) [Priestley] Let $\underline{\mathbf{L}} = \langle \{0,1\}; \vee, \wedge\rangle$; then $\mathcal{L} = \mathrm{ISP}(\underline{\mathbf{L}})$ is the class of distributive lattices. Show that $\underset{\sim}{\mathbf{L}} = \langle \{0,1\}; 0, 1, \leqslant, \mathcal{T}\rangle$ yields a duality on $\mathcal{L}$.

(3) [Hofmann–Mislove–Stralka] Let $\underline{\mathbf{S}} = \langle \{0,1\}; \wedge\rangle$; then $\mathcal{S} = \mathrm{ISP}(\underline{\mathbf{S}})$ is the class of meet semilattices. Show that $\underset{\sim}{\mathbf{S}} = \langle \{0,1\}; \wedge, 0, 1, \mathcal{T}\rangle$ yields a duality on $\mathcal{S}$.

(4) [Pontryagin] Let $\underline{\mathbf{Z}}_m = \langle \mathbb{Z}_m; +, ^{-}, 0\rangle$; then $\mathcal{A}_m = \mathrm{ISP}(\underline{\mathbf{Z}}_m)$ is the class of abelian groups of exponent m. Show that $\underset{\sim}{\mathbf{Z}} = \langle \mathbb{Z}_m; +, ^{-}, 0, \mathcal{T}\rangle$ yields a duality on $\mathcal{A}_m$.