Lecture 2: An invitation to natural dualities

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Outline

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Examples of natural dualities
- Boolean algebras – Stone
- Distributive lattices – Priestley
- Abelian groups – Pontryagin

Natural dualities: the basics

Duality theorems

Boolean algebras — Stone duality

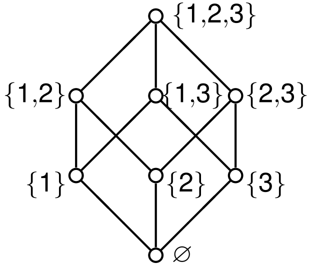
Boolean algebras

Boolean spaces

(i.e., compact, Hausdorff and a basis of clopen sets)

Boolean algebras — Stone duality

Boolean algebras



Boolean spaces

(i.e., compact, Hausdorff and a basis of clopen sets)

Boolean algebras — Stone duality

Boolean algebras

Boolean algebra of all
finite or cofinite subsets of $\mathbb{N}$

Boolean spaces

(i.e., compact, Hausdorff and
a basis of clopen sets)

$$\begin{array}{cccccc} \circ & \circ & \circ & \circ & \cdots & \circ \\ 1 & 2 & 3 & 4 & & \infty \end{array}$$

Boolean algebras

Countable atomless
Boolean algebra

$\mathbf{F}_{\mathcal{B}}(\omega)$

Boolean spaces

(i.e., compact, Hausdorff and
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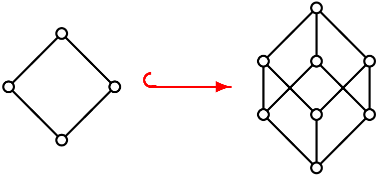
Boolean algebras — Stone duality

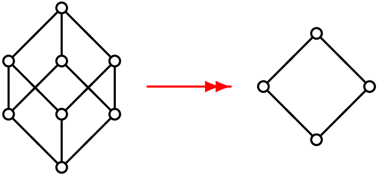
Boolean algebras



Boolean spaces

Boolean algebras — Stone duality

Boolean algebras



Boolean spaces

Boolean algebras — Stone duality

Boolean algebras

Boolean spaces

$\mathcal{B} = \mathrm{ISP}(\underline{\mathbf{B}})$, where

$\underline{\mathbf{B}} = \langle\{0,1\}; \vee, \wedge, ', 0, 1\rangle$

$D(\mathbf{A}) := \mathcal{B}(\mathbf{A}, \underline{\mathbf{B}}) \leqslant \underset{\sim}{\mathbf{B}}^{A}$

$\mathcal{Z} = \mathrm{IS_cP^{+}}(\underset{\sim}{\mathbf{B}})$, where

$\underset{\sim}{\mathbf{B}} = \langle\{0,1\}; \mathcal{T}\rangle$

$E(\mathbf{X}) := \mathcal{Z}(\mathbf{X}, \underset{\sim}{\mathbf{B}}) \leqslant \mathbf{B}^{X}$

Bounded distributive lattices — Priestley duality

Bounded distributive lattices

Priestley spaces

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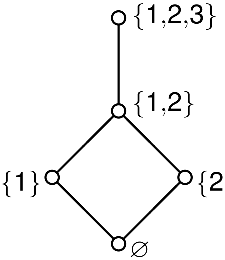
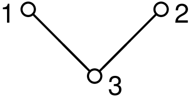


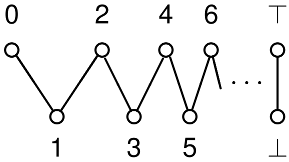


Bounded distributive lattices — Priestley duality

Bounded distributive lattices

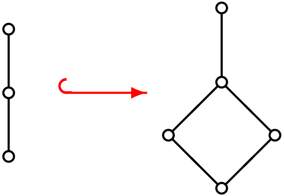
?

Priestley spaces

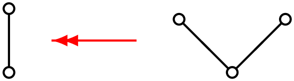


Bounded distributive lattices — Priestley duality

Bounded distributive lattices



Priestley spaces



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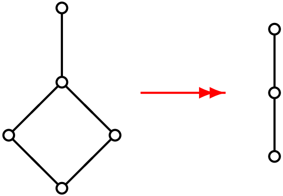

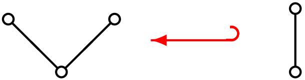

Bounded distributive lattices

$\mathcal{D} = \mathsf{ISP}(\underline{\mathbf{D}})$, where

$\underline{\mathbf{D}} = \langle \{0,1\}; \vee, \wedge, 0, 1\rangle$

$D(\mathbf{A}) := \mathcal{D}(\mathbf{A}, \underline{\mathbf{D}}) \leqslant \underset{\sim}{\mathbf{D}}^{A}$

Priestley spaces

$\mathcal{P} = \mathsf{IS_cP^+}(\underset{\sim}{\mathbf{D}})$, where

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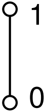
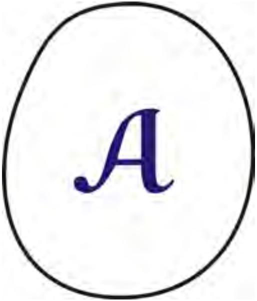


$E(\mathbf{X}) := \mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{D}}) \leqslant \underline{\mathbf{D}}^{X}$

Abelian groups — Pontryagin duality

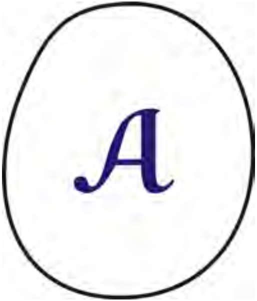


Abelian groups

Compact top. abelian groups

(i.e., compact, Hausdorff and $\cdot$ and $^{-1}$ continuous)

Abelian groups — Pontryagin duality



Abelian groups

$\mathbf{Z} = \langle Z; +, -, 0\rangle$

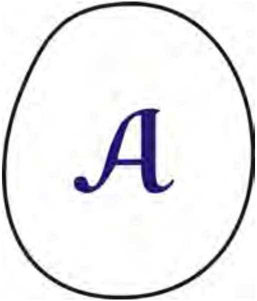
The integers

Compact top. abelian groups

$\underset{\sim}{\mathbf{T}} = \langle T; \cdot, {}^{-1}, 1, \mathcal{T}\rangle$

The circle group

Abelian groups — Pontryagin duality



Abelian groups

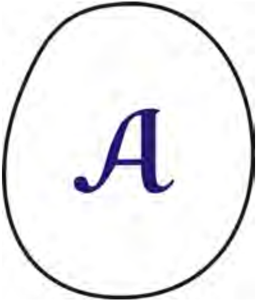
$\mathbf{Z}_n = \langle \mathbb{Z}_n; \oplus_n, \ominus_n, 0 \rangle$

The integers modulo n

Compact top. abelian groups

$\mathbf{Z}_n^{\mathcal{T}} = \langle \mathbb{Z}_n; \oplus_n, \ominus_n, 0, \mathcal{T} \rangle$

Abelian groups — Pontryagin duality



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$$\mathbf{A} \hookrightarrow \mathbf{B}$$

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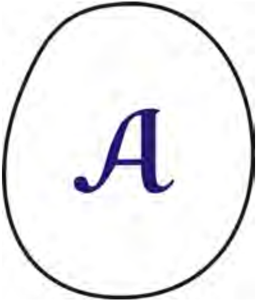

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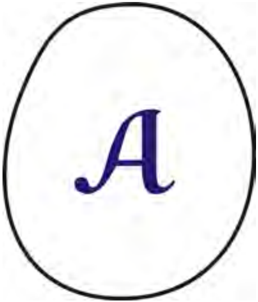

Abelian groups

$\mathcal{A} = \mathrm{ISP}(\underline{\mathbf{T}})$, where

$\underline{\mathbf{T}} = \langle T; \cdot, {}^{-1}, 1 \rangle$

and $T := \{z \in \mathbb{C} : |x| = 1\}$

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Compact top. abelian groups

$\mathcal{X} = \mathrm{IS_cP^+}(\underset{\sim}{\mathbf{T}})$, where

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Generalising to natural dualities: “why bother?"

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- Some dualities have the powerful property of being “logarithmic”—they turn products into sums; e.g., in both $\mathcal{B}$ and $\mathcal{D}$ we have $D(\mathbf{A} \times \mathbf{B}) \cong D(\mathbf{A}) \mathbin{\dot\cup} D(\mathbf{B})$.

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  1. (1) free algebras in $\mathcal{A}$ are easily described via their duals in $\mathcal{X}$,
  2. (2) while a coproduct $\mathbf{A} * \mathbf{B}$ is often difficult to describe, its dual, $D(\mathbf{A} * \mathbf{B})$, is simply the cartesian product $D(\mathbf{A}) \times D(\mathbf{B})$,

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  (3) congruence lattices in $\mathcal{A}$ may be studied by looking at lattices of closed substructures in $\mathcal{X}$,
  (4) injective algebras in $\mathcal{A}$ may be characterised by first studying projective structures in $\mathcal{X}$,
  (5) algebraically closed and existentially closed algebras may be described via their duals.

Some observations on $\mathcal{B}$, $\mathfrak{D}$ and $\mathcal{A}$

For the functors D and E to be well defined, we need the algebras $\underline{\mathbf{B}}$, $\underline{\mathbf{D}}$ and $\underline{\mathbf{T}}$ and the corresponding topological structures $\utilde{\mathbf{B}}$, $\utilde{\mathbf{D}}$ and $\utilde{\mathbf{T}}$ to be compatible.

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Let $\underline{\mathbf{M}} = \langle M; F\rangle$ be one of $\underline{\mathbf{B}}$, $\underline{\mathbf{D}}$ and $\underline{\mathbf{T}}$, and let $\underset{\sim}{\mathbf{M}} = \langle M; G, R, \mathcal{T}\rangle$ be the corresponding topological structure, $\underset{\sim}{\mathbf{B}}$, $\underset{\sim}{\mathbf{D}}$ or $\underset{\sim}{\mathbf{T}}$.

Define $\mathcal{A} := \mathsf{ISP}(\underline{\mathbf{M}})$ and $\mathcal{X} := \mathsf{IS_cP^+}(\underset{\sim}{\mathbf{M}})$, and let $\mathbf{A} \in \mathcal{A}$ and $\mathbf{X} \in \mathcal{X}$.

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Since we define $D(\mathbf{A}) := \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ and $E(\mathbf{X}) := \mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$, in order to have $D(\mathbf{A}) \in \mathrm{IS_cP^+}(\underset{\sim}{\mathbf{M}})$ and $E(\mathbf{X}) \in \mathrm{ISP}(\underline{\mathbf{M}})$, we need

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Some observations

Let $\underline{\mathbf{M}} = \langle M; F \rangle$, let $\underset{\sim}{\mathbf{M}} = \langle M; G, R, \mathcal{T} \rangle$, define $\mathcal{A} := \mathsf{ISP}(\underline{\mathbf{M}})$ and $\mathcal{X} := \mathsf{IS_cP^+}(\underset{\sim}{\mathbf{M}})$, and let $\mathbf{A} \in \mathcal{A}$ and $\mathbf{X} \in \mathcal{X}$. We need $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}$ to be compatible in such a way that

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  - $\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ will be topologically closed in M^A, provided the topology on M is Hausdorff and the operations in F are continuous.

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  - $\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ will be closed under the operations in G provided each (n-ary) $g \in G$ is a homomorphism from $\underline{\mathbf{M}}^n$ to $\underline{\mathbf{M}}$.

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  - $\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ will be topologically closed in M^A, provided the topology on M is Hausdorff and the operations in F are continuous. (If $\underset{\sim}{\mathbf{M}}$ is compact, then so is $\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$.)
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    - each (n-ary) $g \in G$ is a homomorphism from $\underline{\mathbf{M}}^n$ to $\underline{\mathbf{M}}$,
    - each (n-ary) relation $r \in R$ is a subuniverse of $\underline{\mathbf{M}}^n$, and
    - each operation in F is continuous.

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Some observations

Let $\underline{\mathbf{M}} = \langle M; F \rangle$, let $\underset{\sim}{\mathbf{M}} = \langle M; G, R, \mathcal{T} \rangle$, define $\mathcal{A} := \mathbb{ISP}(\underline{\mathbf{M}})$ and $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$, and let $\mathbf{A} \in \mathcal{A}$ and $\mathbf{X} \in \mathcal{X}$. We need $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}$ to be compatible in such a way that

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When these highlighted conditions hold, we say that g and r are compatible with or algebraic over $\underline{\mathbf{M}}$.

Outline

Natural dualities: alter egos

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- For each structure $\mathbf{X}$ in $\mathcal{X}$, the underlying set of $E(\mathbf{X})$ is the set $\hom(\mathbf{X}, \underset{\sim}{\mathbf{M}})$ of all continuous homomorphisms from $\mathbf{X}$ into $\underset{\sim}{\mathbf{M}}$, and $E(\mathbf{X})$ is a subalgebra of $\underline{\mathbf{M}}^X$.

Natural dualities: embeddings

Natural embeddings

For all $\mathbf{A} \in \mathcal{A}$ and $\mathbf{X} \in \mathcal{X}$, there are embeddings

$$e_{\mathbf{A}} \colon \mathbf{A} \to ED(\mathbf{A}) = \mathcal{X}(\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}}), \utilde{\mathbf{M}}), \quad \text{given by}$$
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These embeddings yield natural transformations

$$e \colon \mathrm{id}_{\mathcal{A}} \to ED \qquad \text{and} \qquad \varepsilon \colon \mathrm{id}_{\mathcal{X}} \to DE,$$

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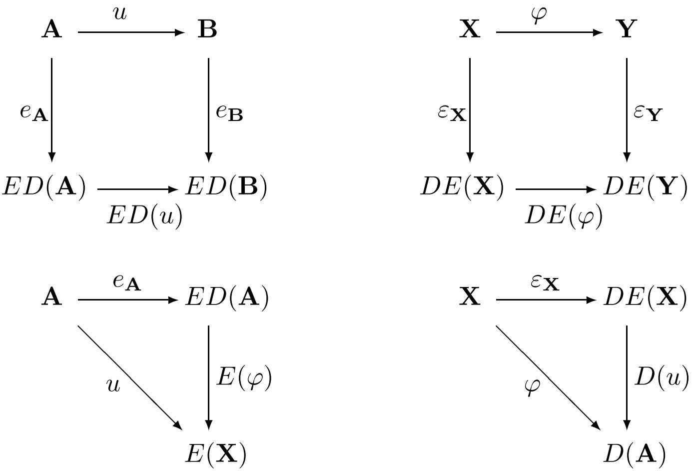
A dual adjunction

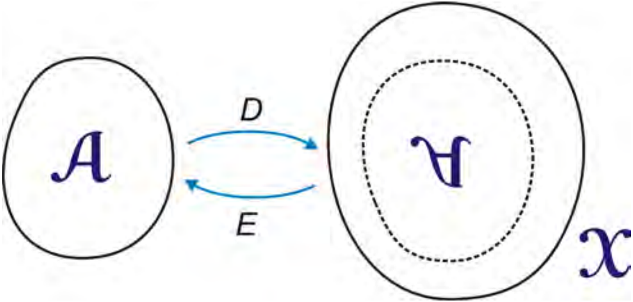


- For $u\colon \mathbf{A} \to \mathbf{B}$ and $\varphi\colon \mathbf{X} \to \mathbf{Y}$, the two squares commute.
- $\mathcal{A}(\mathbf{A}, E(\mathbf{X})) \cong \mathfrak{X}(\mathbf{X}, D(\mathbf{A}))$ via the triangles:

$$u = E(D(u) \circ \varepsilon_{\mathbf{X}}) \circ e_{\mathbf{A}} \text{ and } \varphi = D(E(\varphi) \circ e_{\mathbf{A}}) \circ \varepsilon_{\mathbf{X}}.$$

Duality

If $e_{\mathbf{A}} : \mathbf{A} \to ED(\mathbf{A})$ is surjective and therefore an isomorphism, for all $\mathbf{A}$ in $\mathcal{A}$, then we say that $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$ (or that $\underset{\sim}{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$).



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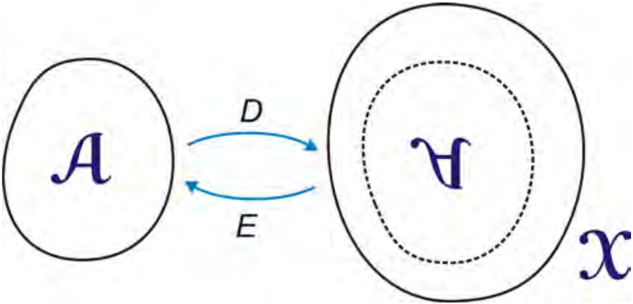
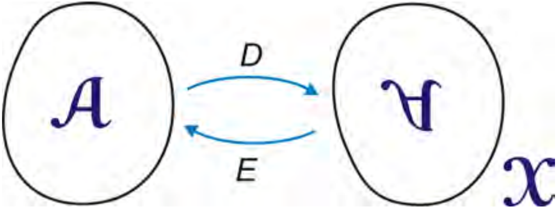


Equivalently, $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$ if the dual adjunction $\langle D, E, e, \varepsilon \rangle$ is a dual category equivalence between $\mathcal{A}$ and a full subcategory of $\mathcal{X}$.

Full duality

If, in addition, $\varepsilon_{\mathbf{X}} \colon \mathbf{X} \to DE(\mathbf{X})$ is a surjection and therefore an isomorphism, for all $\mathbf{X}$ in $\mathcal{X}$, then $\underset{\sim}{\mathbf{M}}$ yields a full duality on $\mathcal{A}$ (or $\underset{\sim}{\mathbf{M}}$ fully dualises $\underline{\mathbf{M}}$).



Equivalently, $\underset{\sim}{\mathbf{M}}$ yields a full duality on $\mathcal{A}$ if the dual adjunction $\langle D, E, e, \varepsilon \rangle$ is a dual category equivalence between $\mathcal{A}$ and $\mathcal{X}$.

Embeddings, injectivity and strong duality

Let $\underset{\sim}{\mathbf{M}}$ be any alter ego of an algebra $\underline{\mathbf{M}}$, and let

$$D\colon \mathcal{A} \to \mathfrak{X} \qquad \text{and} \qquad E\colon \mathfrak{X} \to \mathcal{A}$$

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Strong duality

If $\underset{\sim}{\mathbf{M}}$ fully dualises $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}$ is injective in $\mathcal{X}$ (so that surjections in $\mathcal{A}$ correspond to embeddings in $\mathcal{X}$), we say that $\underset{\sim}{\mathbf{M}}$ yields a strong duality on $\mathcal{A}$ (or that $\underset{\sim}{\mathbf{M}}$ strongly dualises $\underline{\mathbf{M}}$).

Further examples

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- The two-element implication algebra $\mathbf{I} := \langle \{0, 1\}; \rightarrow \rangle$ does not admit a natural duality. [Davey, Werner 1980]

Outline

For duality, relations will do

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Recall that to prove that $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$, it remains to show that

- for all $\mathbf{A} \in \mathcal{A}$, the evaluation maps $e_{\mathbf{A}}$, for $a \in A$, are the only $\mathcal{X}$-morphisms from $\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ to $\underset{\sim}{\mathbf{M}}$.

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Lemma (2.1.2)

Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$, define $\underset{\sim}{\mathbf{M}}' = \langle M; R', \mathcal{T}\rangle$ where

$$R' := R \cup \{\mathrm{graph}(h) \mid h \in G \cup H\}$$

Then $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$ if and only if $\underset{\sim}{\mathbf{M}}'$ does.

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- Thus, as far as obtaining a duality is concerned, we can restrict our attention to purely relational alter egos.

Duals of free algebras

- Given a non-empty set S, the set

  $\mathbf{F}_{\underline{\mathbf{M}}}(S) = \{t\colon M^S \to M \mid t \text{ is an } S\text{-ary term function on } \underline{\mathbf{M}}\}$

  is the free S-generated algebra in $\mathcal{A}$ (the projections $\pi_s\colon M^S \to M$, for $s \in S$, are the free generators).

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The interpolation condition (IC)

Let $\mathcal{A}_{\text{fin}}$ and $\mathcal{X}_{\text{fin}}$ consist of the finite members of $\mathcal{A}$ and $\mathcal{X}$.

Lemma (2.2.5)

The following are equivalent:

(i) (IC) *for each $n \in \mathbb{N}$ and each substructure $\mathbf{X}$ of $\underset{\sim}{\mathbf{M}}^n$, every morphism $\alpha \colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ extends to a term function $t \colon M^n \to M$ of the algebra $\underline{\mathbf{M}}$,*

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This is achievable provided $\underset{\sim}{\mathbf{M}}$ enjoys some degree of finiteness.

The Second Duality Theorem

If $\underset{\sim}{\mathbf{M}} = \langle M; G, R, \mathcal{T} \rangle$, that is, the type of $\underset{\sim}{\mathbf{M}}$ includes no partial operations, then we call $\underset{\sim}{\mathbf{M}}$ a total structure.

Theorem (2.2.7 Second Duality Theorem)

Assume that $\underset{\sim}{\mathbf{M}}$ is a total structure with R finite. If (IC) *holds, then $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$ and is injective in $\mathcal{X}$.*

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This result is rather surprising.

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- It gives us simple finitary conditions which yield both a dual adjunction between the categories $\mathcal{A}$ and $\mathcal{X}$ and a topological representation of every algebra in $\mathcal{A}$,
- but it requires us to do no category theory and no topology!

Priestley duality via the Second Duality Theorem

Recall that

- $\underline{\mathbf{D}} = \langle \{0, 1\}; \vee, \wedge, 0, 1 \rangle$ is the two-element bounded lattice,
- $\underset{\sim}{\mathbf{D}} = \langle \{0, 1\}; \leqslant, \mathcal{T} \rangle$ is the two-element chain endowed with the discrete topology.

Theorem (Half of Priestley duality)

$\underset{\sim}{\mathbf{D}}$ yields a duality on the class $\mathcal{D} := \mathrm{ISP}(\underline{\mathbf{D}})$ of bounded distributive lattices, i.e., $e_{\mathbf{A}} \colon \mathbf{A} \to ED(\mathbf{A})$ is an isomorphism, for all $\mathbf{A} \in \mathcal{D}$.

Proof.

We will prove that (IC) holds. Let $\mathbf{X}$ be a substructure of $\underset{\sim}{\mathbf{D}}^n$ and let $\varphi \colon \mathbf{X} \to \underset{\sim}{\mathbf{D}}$ be a morphism, i.e., φ is order-preserving.

[We need to find a term function $t \colon \{0, 1\}^n \to \{0, 1\}$ on $\underline{\mathbf{D}}$ such that $t(x) = \varphi(x)$, for all $x \in X$.]

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If $\varphi^{-1}(1) = \varnothing$, then define $t(v_1, \ldots, v_n) = 0$, and
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Further applications of the Second Duality Theorem

Some exercises for you. In each case, prove that (IC) holds.

(1) [Stone] Let $\underline{\mathbf{B}} = \langle \{0,1\}; \vee, \wedge, ', 0, 1 \rangle$; then $\mathcal{B} = \mathsf{ISP}(\underline{\mathbf{B}})$ is the class of Boolean algebras. Show that $\underset{\sim}{\mathbf{B}} = \langle \{0,1\}; \mathcal{T} \rangle$ yields a duality on $\mathcal{B}$.

(2) [Priestley] Let $\underline{\mathbf{L}} = \langle \{0,1\}; \vee, \wedge \rangle$; then $\mathcal{L} = \mathsf{ISP}(\underline{\mathbf{L}})$ is the class of distributive lattices. Show that $\underset{\sim}{\mathbf{L}} = \langle \{0,1\}; 0, 1, \leqslant, \mathcal{T} \rangle$ yields a duality on $\mathcal{L}$.

(3) [Hofmann–Mislove–Stralka] Let $\underline{\mathbf{S}} = \langle \{0,1\}; \wedge \rangle$; then $\mathcal{S} = \mathsf{ISP}(\underline{\mathbf{S}})$ is the class of meet semilattices. Show that $\underset{\sim}{\mathbf{S}} = \langle \{0,1\}; \wedge, 0, 1, \mathcal{T} \rangle$ yields a duality on $\mathcal{S}$.

(4) [Pontryagin] Let $\underline{\mathbf{Z}}_m = \langle \mathbb{Z}_m; +, ^-, 0 \rangle$; then $\mathcal{A}_m = \mathsf{ISP}(\underline{\mathbf{Z}}_m)$ is the class of abelian groups of exponent m. Show that $\underset{\sim}{\mathbf{Z}} = \langle \mathbb{Z}_m; +, ^-, 0, \mathcal{T} \rangle$ yields a duality on $\mathcal{A}_m$.