Lecture 2: An invitation to natural dualities

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Examples of natural dualities

Natural dualities: the basics

Duality theorems
Examples of natural dualities
  Boolean algebras – Stone
  Distributive lattices – Priestley
  Abelian groups – Pontryagin

Natural dualities: the basics

Duality theorems
Boolean algebras — Stone duality

Boolean algebras

Boolean spaces

(i.e., compact, Hausdorff and a basis of clopen sets)
Boolean algebras — Stone duality

Boolean algebras

Boolean spaces

(i.e., compact, Hausdorff and a basis of clopen sets)
Boolean algebras — Stone duality

Boolean algebra of all finite or cofinite subsets of \( \mathbb{N} \)

Boolean spaces
(i.e., compact, Hausdorff and a basis of clopen sets)
Boolean algebras — Stone duality

Boolean algebras and Boolean spaces
(i.e., compact, Hausdorff and a basis of clopen sets)

Countable atomless Boolean algebra
\( F_B(\omega) \)

| 0 | 1/9 | 2/9 | 1/3 | 2/3 | 7/9 | 8/9 | 1 |
Boolean algebras — Stone duality

Boolean algebras

Boolean spaces
Boolean algebras — Stone duality

Boolean algebras

Boolean spaces
Boolean algebras — Stone duality

\[ \mathcal{B} = \text{ISP}(\mathcal{B}), \text{ where} \]
\[ \mathcal{B} = \langle \{0, 1\}; \lor, \land, ', 0, 1 \rangle \]
\[ D(A) := \mathcal{B}(A, \mathcal{B}) \leq \mathcal{B}^A \]

\[ \mathcal{Z} = \text{IS}_c \mathcal{P}^+(\mathcal{B}), \text{ where} \]
\[ \mathcal{Z} = \langle \{0, 1\}; \mathcal{I} \rangle \]
\[ E(X) := \mathcal{Z}(X, \mathcal{B}) \leq \mathcal{B}^X \]
Bounded distributive lattices — Priestley duality
Bounded distributive lattices — Priestley duality

Bounded distributive lattices

Priestley spaces
Bounded distributive lattices — Priestley duality

Bounded distributive lattices

?  

 Priestley spaces

\[
\begin{align*}
\text{0} & \quad 2 & \quad 4 & \quad 6 & \quad \top \\
& \quad & \quad & \quad & \\
1 & \quad 3 & \quad 5 & \quad \bot
\end{align*}
\]
Bounded distributive lattices — Priestley duality

Bounded distributive lattices

Priestley spaces
Bounded distributive lattices — Priestley duality

Bounded distributive lattices

Priestley spaces
Bounded distributive lattices — Priestley duality

\[ \mathcal{D} = \text{ISP}(D), \text{ where} \]
\[ D = \langle \{0, 1\}; \lor, \land, 0, 1 \rangle \]
\[ D(A) := \mathcal{D}(A, D) \leq D^A \]

\[ \mathcal{P} = \text{ISP}^+(\mathcal{D}), \text{ where} \]
\[ \mathcal{D} = \langle \{0, 1\}; \leq, \cong \rangle \]
\[ E(X) := \mathcal{P}(X, \mathcal{D}) \leq D^X \]
Abelian groups — Pontryagin duality

Abelian groups

Compact top. abelian groups
(i.e., compact, Hausdorff and \( \cdot \) and \( \cdot^{-1} \) continuous)
Abelian groups — Pontryagin duality

Z = \langle \mathbb{Z}; +, −, 0 \rangle
The integers

\mathcal{T} = \langle \mathcal{T}; \cdot, −1, 1, \mathcal{T} \rangle
The circle group
Abelian groups — Pontryagin duality

\[ \mathbb{Z}_n = \langle \mathbb{Z}_n; \oplus_n, \ominus_n, 0 \rangle \]

The integers modulo \( n \)

\[ \mathbb{Z}_n^\mathcal{T} = \langle \mathbb{Z}_n; \oplus_n, \ominus_n, 0, \mathcal{T} \rangle \]

Compact top. abelian groups
Abelian groups — Pontryagin duality

Abelian groups

Compact top. abelian groups

A ↦ B

D(A) ↦ D(B)
Abelian groups — Pontryagin duality

Abelian groups

$A \rightarrow B$

Compact top. abelian groups

$D(A) \rightarrow D(B)$
Abelian groups — Pontryagin duality

\[ \mathcal{A} = \text{ISP}(T), \text{ where} \]
\[ T = \langle T; \cdot, -1, 1 \rangle \]
and \[ T := \{ z \in \mathbb{C} : |x| = 1 \} \]

\[ D(\mathcal{A}) := \mathcal{A}(\mathcal{A}, T) \leq T^A \]

\[ \mathcal{X} = \text{IS}_c^+P(T), \text{ where} \]
\[ T = \langle T; \cdot, -1, 1, \mathcal{T} \rangle \]
\[ E(X) := \mathcal{X}(X, T) \leq T^X \]
Generalising to natural dualities: “why bother?”
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Let \( M = \langle M; F \rangle \) be one of \( B, D \) and \( T \), and let \( \mathcal{M} = \langle M; G, R, \mathcal{J} \rangle \) be the corresponding topological structure, \( \mathcal{B}, \mathcal{D} \) or \( \mathcal{T} \).
Generalising to natural dualities: “why bother?"

Let $\mathbf{M} = \langle M; F \rangle$ be one of $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{T}$, and let $\mathbf{M} = \langle M; G, R, \mathcal{J} \rangle$ be the corresponding topological structure, $\mathcal{B}$, $\mathcal{D}$ or $\mathcal{T}$.

- A duality for $\mathcal{A} := \text{ISP}(\mathbf{M})$ gives a uniform way to represent each algebra $\mathbf{A} \in \mathcal{A}$ as an algebra of continuous functions.
Generalising to natural dualities: “why bother?”

Let $\mathcal{M} = \langle M; F \rangle$ be one of $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{T}$, and let $\mathcal{M} = \langle M; G, R, T \rangle$ be the corresponding topological structure, $\mathcal{B}$, $\mathcal{D}$ or $\mathcal{T}$.

- A duality for $\mathcal{A} := \text{ISP}(\mathcal{M})$ gives a uniform way to represent each algebra $A \in \mathcal{A}$ as an algebra of continuous functions.
- If we have a full duality and have axiomatised the class $\mathcal{X} := \text{ISP}^+(\mathcal{M})$, we can find examples of algebras in $\mathcal{A}$ by simply constructing objects in $\mathcal{X}$. 
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- A duality for $\mathcal{A} := \text{ISP}(\mathbf{M})$ gives a uniform way to represent each algebra $\mathbf{A} \in \mathcal{A}$ as an algebra of continuous functions.
- If we have a full duality and have axiomatised the class $\mathbf{X} := \text{ISP}^+(\mathbf{M})$, we can find examples of algebras in $\mathcal{A}$ by simply constructing objects in $\mathbf{X}$.
- Some dualities have the powerful property of being “logarithmic”—they turn products into sums; e.g., in both $\mathbf{B}$ and $\mathbf{D}$ we have $D(\mathbf{A} \times \mathbf{B}) \cong D(\mathbf{A}) \cup D(\mathbf{B})$. 
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Generalising to natural dualities: “why bother?”

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Algebraic questions in $\mathcal{A}$ can be answered by translating them into (often simpler) questions in $\mathcal{X}$. For example,

(1) free algebras in $\mathcal{A}$ are easily described via their duals in $\mathcal{X}$,

(2) while a coproduct $A \ast B$ is often difficult to describe, its dual, $D(A \ast B)$, is simply the cartesian product $D(A) \times D(B)$,
Algebraic questions in $\mathcal{A}$ can be answered by translating them into (often simpler) questions in $\mathcal{X}$. For example,

1. free algebras in $\mathcal{A}$ are easily described via their duals in $\mathcal{X}$,
2. while a coproduct $A \ast B$ is often difficult to describe, its dual, $D(A \ast B)$, is simply the cartesian product $D(A) \times D(B)$,
3. congruence lattices in $\mathcal{A}$ may be studied by looking at lattices of closed substructures in $\mathcal{X}$,
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3. congruence lattices in $\mathcal{A}$ may be studied by looking at lattices of closed substructures in $\mathcal{X}$,
4. injective algebras in $\mathcal{A}$ may be characterised by first studying projective structures in $\mathcal{X}$,
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2. while a coproduct $A \ast B$ is often difficult to describe, its dual, $D(A \ast B)$, is simply the cartesian product $D(A) \times D(B)$,
3. congruence lattices in $\mathcal{A}$ may be studied by looking at lattices of closed substructures in $\mathcal{X}$,
4. injective algebras in $\mathcal{A}$ may be characterised by first studying projective structures in $\mathcal{X}$,
5. algebraically closed and existentially closed algebras may be described via their duals.
Some observations on $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{A}$

For the functors $D$ and $E$ to be well defined, we need the algebras $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{T}$ and the corresponding topological structures $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{T}$ to be compatible.
Some observations on $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{A}$

For the functors $D$ and $E$ to be well defined, we need the algebras $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{T}$ and the corresponding topological structures $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{T}$ to be compatible.

Let $\mathcal{M} = \langle M; F \rangle$ be one of $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{T}$, and let $\mathcal{M} \sim = \langle M; G, R, \mathcal{T} \rangle$ be the corresponding topological structure, $\mathcal{B}$, $\mathcal{D}$ or $\mathcal{T}$.

Define $\mathcal{A} := \text{ISP}(\mathcal{M})$ and $\mathcal{X} := \text{IS}_cP^+(\mathcal{M})$, and let $A \in \mathcal{A}$ and $X \in \mathcal{X}$.
Some observations on $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{A}$

For the functors $D$ and $E$ to be well defined, we need the algebras $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{T}$ and the corresponding topological structures $\mathcal{B} \sim$, $\mathcal{D} \sim$ and $\mathcal{T} \sim$ to be compatible.

Let $\underline{M} = \langle M; F \rangle$ be one of $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{T}$, and let $\underline{\sim} = \langle M; G, R, T \rangle$ be the corresponding topological structure, $\mathcal{B} \sim$, $\mathcal{D} \sim$ or $\mathcal{T} \sim$.

Define $\mathcal{A} := \text{ISP}(\underline{M})$ and $\mathcal{X} := \text{IS}_c\text{P}^+(\underline{M})$, and let $\underline{A} \in \mathcal{A}$ and $\underline{X} \in \mathcal{X}$.

Since we define $D(\underline{A}) := \mathcal{A}(\underline{A}, \underline{M})$ and $E(\underline{X}) := \mathcal{X}(\underline{X}, \underline{M})$, in order to have $D(\underline{A}) \in \text{IS}_c\text{P}^+(\underline{M})$ and $E(\underline{X}) \in \text{ISP}(\underline{M})$, we need

- $\mathcal{A}(\underline{A}, \underline{M})$ to be a topologically closed substructure of $\underline{M}^A$, and
Some observations on $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{A}$

For the functors $D$ and $E$ to be well defined, we need the algebras $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{T}$ and the corresponding topological structures $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{T}$ to be compatible.

Let $\underline{M} = \langle M; F \rangle$ be one of $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{T}$, and let $\underline{M} \sim = \langle M; G, R, \mathcal{T} \rangle$ be the corresponding topological structure, $\mathcal{B}$, $\mathcal{D}$ or $\mathcal{T}$.

Define $\mathcal{A} := \text{ISP}(\underline{M})$ and $\mathcal{X} := \text{IS}_c \mathcal{P}^+(\underline{M})$, and let $A \in \mathcal{A}$ and $X \in \mathcal{X}$.

Since we define $D(A) := \mathcal{A}(A, \underline{M})$ and $E(X) := \mathcal{X}(X, \underline{M})$, in order to have $D(A) \in \text{IS}_c \mathcal{P}^+(\underline{M})$ and $E(X) \in \text{ISP}(\underline{M})$, we need

- $\mathcal{A}(A, \underline{M})$ to be a topologically closed substructure of $\underline{M}^A$,
  and
- $\mathcal{X}(X, \underline{M})$ to be a subalgebra of $\underline{M}^X$. 
Some observations

Let $\mathbf{M} = \langle M; F \rangle$, let $\mathbf{M} = \langle M; G, R, \top \rangle$, define $\mathcal{A} := \text{ISP}(\mathbf{M})$ and $\mathcal{X} := \text{IS}_{c}P^{+}(\mathbf{M})$, and let $A \in \mathcal{A}$ and $X \in \mathcal{X}$. We need $\mathbf{M}$ and $\mathbf{M}$ to be compatible in such a way that

- $\mathcal{A}(A, \mathbf{M})$ is a topologically closed substructure of $\mathbf{M}^{A}$, and
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Let $\underline{M} = \langle M; F \rangle$, let $\underline{M} = \langle M; G, R, T \rangle$, define $\mathcal{A} := \text{ISP}(\underline{M})$ and $\mathcal{X} := \text{IS}_{c}P^{+}(\underline{M})$, and let $\mathcal{A} \in \mathcal{A}$ and $\mathcal{X} \in \mathcal{X}$. We need $\underline{M}$ and $\underline{M}$ to be compatible in such a way that

- $\mathcal{A}(\mathcal{A}, \underline{M})$ is a topologically closed substructure of $\underline{M}^\mathcal{A}$, and
- $\mathcal{X}(\mathcal{X}, \underline{M})$ is a subalgebra of $\underline{M}^\mathcal{X}$.

$\triangleright$ $\mathcal{A}(\mathcal{A}, \underline{M})$ will be topologically closed in $M^\mathcal{A}$, provided the topology on $M$ is Hausdorff and the operations in $F$ are continuous.
Some observations

Let $\mathbf{M} = \langle M; F \rangle$, let $\mathbf{\sim} = \langle M; G, R, \mathcal{T} \rangle$, define $\mathcal{A} := \text{ISP}(\mathbf{M})$ and $\mathcal{X} := \text{IScP}^+(\mathbf{\sim})$, and let $A \in \mathcal{A}$ and $X \in \mathcal{X}$. We need $\mathbf{M}$ and $\mathbf{\sim}$ to be compatible in such a way that

- $\mathcal{A}(A, \mathbf{M})$ is a topologically closed substructure of $\mathbf{\sim}^A$, and
- $\mathcal{X}(X, \mathbf{\sim})$ is a subalgebra of $\mathbf{M}^X$.

$\mathcal{A}(A, \mathbf{M})$ will be topologically closed in $M^A$, provided the topology on $M$ is Hausdorff and the operations in $F$ are continuous. (If $\mathbf{M}$ is compact, then so is $\mathcal{A}(A, \mathbf{M})$.)
Some observations

Let $\mathbf{M} = \langle M; F \rangle$, let $\mathbf{M} = \langle M; G, R, \mathcal{I} \rangle$, define $\mathcal{A} := \text{ISP}(\mathbf{M})$ and $\mathcal{X} := \text{IScP}^+(\mathbf{M})$, and let $\mathbf{A} \in \mathcal{A}$ and $\mathbf{X} \in \mathcal{X}$. We need $\mathbf{M}$ and $\mathbf{M}$ to be compatible in such a way that

- $\mathcal{A}(\mathbf{A}, \mathbf{M})$ is a topologically closed substructure of $\mathbf{M}^A$, and
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- $\mathcal{A}(\mathbf{A}, \mathbf{M})$ will be topologically closed in $M^A$, provided the topology on $M$ is Hausdorff and the operations in $F$ are continuous. (If $\mathbf{M}$ is compact, then so is $\mathcal{A}(\mathbf{A}, \mathbf{M})$.)

- $\mathcal{A}(\mathbf{A}, \mathbf{M})$ will be closed under the operations in $G$ provided each $(n$-ary) $g \in G$ is a homomorphism from $\mathbf{M}^n$ to $\mathbf{M}$. 


Some observations

Let $\mathbf{M} = \langle M; F \rangle$, let $\mathbf{M} = \langle M; G, R, \mathcal{T} \rangle$, define $\mathcal{A} := \ISP(\mathbf{M})$ and $\mathcal{X} := \ISC P^+(\mathbf{M})$, and let $\mathbf{A} \in \mathcal{A}$ and $\mathbf{X} \in \mathcal{X}$. We need $\mathbf{M}$ and $\mathbf{M} \sim$ to be compatible in such a way that

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- $\mathcal{A}(\mathbf{A}, \mathbf{M})$ will be topologically closed in $\mathbf{M}^A$, provided the topology on $M$ is Hausdorff and the operations in $F$ are continuous. (If $\mathbf{M}$ is compact, then so is $\mathcal{A}(\mathbf{A}, \mathbf{M})$.)
- $\mathcal{A}(\mathbf{A}, \mathbf{M})$ will be closed under the operations in $G$ provided each ($n$-ary) $g \in G$ is a homomorphism from $\mathbf{M}^n$ to $\mathbf{M}$.
- $\mathcal{X}(\mathbf{X}, \mathbf{M})$ will be a subalgebra of $\mathbf{M}^X$ provided
  - each ($n$-ary) $g \in G$ is a homomorphism from $\mathbf{M}^n$ to $\mathbf{M}$,
  - each ($n$-ary) relation $r \in R$ is a subuniverse of $\mathbf{M}^n$, and
  - each operation in $F$ is continuous.
Let $\mathbf{M} = \langle M; F \rangle$, let $\mathbf{\sim} = \langle M; G, R, T \rangle$, define $\mathcal{A} := \text{ISP}(\mathbf{M})$ and $\mathcal{X} := \text{IScP}^+(\mathbf{M})$, and let $\mathbf{A} \in \mathcal{A}$ and $\mathbf{X} \in \mathcal{X}$. We need $\mathbf{M}$ and $\mathbf{\sim}$ to be compatible in such a way that

- $\mathcal{A}(\mathbf{A}, M)$ is a topologically closed substructure of $M^A$, and
- $\mathcal{X}(\mathbf{X}, M)$ is a subalgebra of $M^X$.

- $\mathcal{A}(\mathbf{A}, M)$ will be topologically closed in $M^A$, provided the topology on $M$ is Hausdorff and the operations in $F$ are continuous. (If $\mathbf{M}$ is compact, then so is $\mathcal{A}(\mathbf{A}, M)$.)
- $\mathcal{A}(\mathbf{A}, M)$ will be closed under the operations in $G$ provided each $(n$-ary $) g \in G$ is a homomorphism from $M^n$ to $M$.
- $\mathcal{X}(\mathbf{X}, M)$ will be a subalgebra of $M^X$ provided
  - each $(n$-ary $) g \in G$ is a homomorphism from $M^n$ to $M$,
  - each $(n$-ary $) relation r \in R$ is a subuniverse of $M^n$, and
  - each operation in $F$ is continuous.
Some observations

Let $M = \langle M; F \rangle$, let $\bar{M} = \langle M; G, R, I \rangle$, define $\mathcal{A} := \text{ISP}(\overline{M})$ and $X := \text{IS}_{c}P^{+}(\overline{M})$, and let $A \in \mathcal{A}$ and $X \in X$. We need $M$ and $\bar{M}$ to be compatible in such a way that

- $\mathcal{A}(A, M)$ is a topologically closed substructure of $\bar{M}^{A}$, and
- $X(X, \bar{M})$ is a subalgebra of $\bar{M}^{X}$.

- $\mathcal{A}(A, M)$ will be topologically closed in $M^{A}$, provided the topology on $M$ is Hausdorff and the operations in $F$ are continuous. (If $\bar{M}$ is compact, then so is $\mathcal{A}(A, M)$.)

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Some observations

Let $\mathbf{M} = \langle M; F \rangle$, let $\mathbf{M} = \langle M; G, R, \mathcal{T} \rangle$, define $\mathcal{A} := \text{ISP}(\mathbf{M})$ and $\mathcal{X} := \text{ISC}^P(\mathbf{M})$, and let $\mathbf{A} \in \mathcal{A}$ and $\mathbf{X} \in \mathcal{X}$. We need $\mathbf{M}$ and $\mathbf{M} \sim$ to be compatible in such a way that

- $\mathcal{A}(\mathbf{A}, \mathbf{M})$ is a topologically closed substructure of $\mathbf{M}^A$, and
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- $\mathcal{A}(\mathbf{A}, \mathbf{M})$ will be closed under the operations in $G$ provided each $(n$-ary) $g \in G$ is a homomorphism from $M^n$ to $M$.
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When these highlighted conditions hold, we say that $g$ and $r$ are compatible with or algebraic over $\mathbf{M}$. 
Examples of natural dualities

Natural dualities: the basics
  Alter egos
  Categories, functors and natural transformations
  The basic definitions: duality, full duality, strong duality
  Further examples

Duality theorems
Generalizing our examples, we start with an algebra $\mathbf{M}$ and wish to find a dual category for the prevariety $\mathcal{A} := \text{ISP}(\mathbf{M})$. 

An alter ego of an algebra $\mathbf{A}$ structure $\mathbf{M} \sim \langle \mathbf{M}; G, H, R, T \rangle$ is an alter ego of $\mathbf{M}$ if it is compatible with $\mathbf{M}$, that is,

- $G$ is a set of operations on $\mathbf{M}$, each of which is a homomorphism with respect to $\mathbf{M}$,
- $H$ is a set of partial operations on $\mathbf{M}$, each of which is a homomorphism with respect to $\mathbf{M}$,
- $R$ is a set of relations on $\mathbf{M}$, each of which is a subuniverse of the appropriate power of $\mathbf{M}$, and
- $T$ is a compact Hausdorff topology on $\mathbf{M}$ with respect to which the operations on $\mathbf{M}$ are continuous.
Natural dualities: alter egos

Generalizing our examples, we start with an algebra $\mathbf{M}$ and wish to find a dual category for the prevariety $\mathcal{A} := \text{ISP}(\mathbf{M})$.

**An alter ego of an algebra**

A structure $\mathfrak{M} = \langle M; G, H, R, T \rangle$ is an alter ego of $\mathbf{M}$ if it is compatible with $\mathbf{M}$, that is,
Natural dualities: alter egos

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- $G$ is a set of operations on $M$, each of which is a homomorphism with respect to $\mathbf{M}$,
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Natural dualities: alter egos

Generalizing our examples, we start with an algebra $\mathbf{M}$ and wish to find a dual category for the prevariety $\mathcal{A} := \text{ISP}(\mathbf{M})$.

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- $R$ is a set of relations on $M$, each of which is a subuniverse of the appropriate power of $\mathbf{M}$, and
- $T$ is a compact Hausdorff topology on $M$ with respect to which the operations on $\mathbf{M}$ are continuous.
Let $\mathcal{M} = \langle M; G, H, R, \mathcal{T} \rangle$ be an alter ego of $\mathcal{M}$. 
Let $\mathcal{M} = \langle M; G, H, R, \mathcal{I} \rangle$ be an alter ego of $\mathcal{M}$.

The categories $\mathcal{A}$ and $\mathcal{X}$

- Define $\mathcal{A} := \text{ISP}(\mathcal{M})$: the algebraic category of interest.
- Define $\mathcal{X} := \text{IScP}^+(\mathcal{M})$: the potential dual category for $\mathcal{A}$. 
Natural dualities: categories and functors

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- Define $\mathcal{X} := \text{IS}_cP^+(\mathcal{M})$: the potential dual category for $\mathcal{A}$.

The contravariant functors $D$ and $E$

- There are natural hom-functors $D: \mathcal{A} \to \mathcal{X}$ and $E: \mathcal{X} \to \mathcal{A}$. 
Natural dualities: categories and functors

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- Define $\mathcal{A} := \text{ISP(}M\text{)}$: the algebraic category of interest.
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The contravariant functors $D$ and $E$

- There are natural hom-functors $D: \mathcal{A} \rightarrow \mathcal{X}$ and $E: \mathcal{X} \rightarrow \mathcal{A}$.
- For each algebra $\mathcal{A}$ in $\mathcal{A}$, the underlying set of $D(\mathcal{A})$ is the set $\text{hom}(\mathcal{A}, \mathcal{M})$ of all homomorphisms from $\mathcal{A}$ into $\mathcal{M}$, and $D(\mathcal{A})$ is a topologically closed substructure of $\mathcal{M}^\mathcal{A}$. 
Natural dualities: categories and functors

Let $\mathcal{M} = \langle M; G, H, R, \tau \rangle$ be an alter ego of $\mathcal{M}$.

The categories $\mathcal{A}$ and $\mathcal{X}$

- Define $\mathcal{A} := ISP(\mathcal{M})$: the algebraic category of interest.
- Define $\mathcal{X} := IS_c P^+(\mathcal{M})$: the potential dual category for $\mathcal{A}$.

The contravariant functors $D$ and $E$

- There are natural hom-functors $D: \mathcal{A} \to \mathcal{X}$ and $E: \mathcal{X} \to \mathcal{A}$.
- For each algebra $A$ in $\mathcal{A}$, the underlying set of $D(A)$ is the set $\text{hom}(A, \mathcal{M})$ of all homomorphisms from $A$ into $\mathcal{M}$, and $D(A)$ is a topologically closed substructure of $\mathcal{M}^A$. 
Natural dualities: categories and functors

Let $\mathcal{M} = \langle M; G, H, R, \tau \rangle$ be an alter ego of $\mathfrak{M}$.

The categories $\mathcal{A}$ and $\mathcal{X}$

- Define $\mathcal{A} := \text{ISP}(\mathcal{M})$: the algebraic category of interest.
- Define $\mathcal{X} := \text{IS}_{c}P^{+}(\mathcal{M})$: the potential dual category for $\mathcal{A}$.

The contravariant functors $D$ and $E$

- There are natural hom-functors $D: \mathcal{A} \rightarrow \mathcal{X}$ and $E: \mathcal{X} \rightarrow \mathcal{A}$.
- For each algebra $\mathfrak{A}$ in $\mathcal{A}$, the underlying set of $D(\mathfrak{A})$ is the set $\text{hom}(\mathfrak{A}, M)$ of all homomorphisms from $\mathfrak{A}$ into $M$, and $D(\mathfrak{A})$ is a topologically closed substructure of $\mathcal{M}^{\mathfrak{A}}$.
- For each structure $\mathfrak{X}$ in $\mathcal{X}$, the underlying set of $E(\mathfrak{X})$ is the set $\text{hom}(\mathfrak{X}, M)$ of all continuous homomorphisms from $\mathfrak{X}$ into $M$, and $E(\mathfrak{X})$ is a subalgebra of $M^{\mathfrak{X}}$. 
Natural dualities: embeddings

Natural embeddings

For all $A \in \mathcal{A}$ and $X \in \mathcal{X}$, there are embeddings

$$e_A : A \rightarrow ED(A) = X(\mathcal{A}(A, M), M),$$

given by

$$\forall a \in A \quad e_A(a) : \mathcal{A}(A, M) \rightarrow M$$

with

$$\forall x \in \mathcal{A}(A, M) \quad e_A(a)(x) := x(a),$$
Natural dualities: embeddings

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For all $A \in \mathcal{A}$ and $X \in \mathcal{X}$, there are embeddings

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and

$$\varepsilon_X : X \to DE(X) = \mathcal{A}(\mathcal{X}(X, \mathcal{M}), \mathcal{M}),$$

given by

$$(\forall x \in X) \ \varepsilon_X(x) : \mathcal{X}(X, \mathcal{M}) \to \mathcal{M} \ with$$

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These embeddings yield natural transformations

$$e : \text{id}_\mathcal{A} \rightarrow ED \quad \text{and} \quad \varepsilon : \text{id}_\mathcal{X} \rightarrow DE,$$

and $\langle D, E, e, \varepsilon \rangle$ is a dual adjunction between $\mathcal{A}$ and $\mathcal{X}$. 
Natural embeddings

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and \( \langle D, E, e, \varepsilon \rangle \) is a dual adjunction between \( \mathcal{A} \) and \( \mathcal{X} \).
A dual adjunction

\[
\begin{array}{ccc}
A & \overset{u}{\longrightarrow} & B \\
& e_A \downarrow & \downarrow e_B \\
ED(A) & \longrightarrow & ED(B) \\
& \downarrow ED(u) & \\
& ED(u) & \\
& \downarrow \downarrow & \\
A & \overset{e_A}{\longrightarrow} & ED(A) \\
& \downarrow u & \downarrow \downarrow \\
& \downarrow E(\varphi) & \\
& \downarrow \downarrow & \\
& E(X) & \\
& \downarrow \downarrow & \\
& \downarrow \downarrow & \\
& \downarrow \downarrow & \\
& D(A) & \\
\end{array}
\]

\[
\begin{array}{ccc}
X & \overset{\varphi}{\longrightarrow} & Y \\
& \varepsilon_X \downarrow & \downarrow \varepsilon_Y \\
DE(X) & \longrightarrow & DE(Y) \\
& \downarrow DE(\varphi) & \\
& \downarrow \downarrow & \\
X & \overset{\varepsilon_X}{\longrightarrow} & DE(X) \\
& \downarrow \varphi & \downarrow \downarrow \\
& \downarrow D(u) & \\
& \downarrow \downarrow & \\
& D(A) & \\
\end{array}
\]

- For \(u: A \to B\) and \(\varphi: X \to Y\), the two squares commute.
- \(\mathcal{A}(A, E(X)) \cong \mathcal{X}(X, D(A))\) via the triangles:

\[
u = E(D(u) \circ \varepsilon_X) \circ e_A \text{ and } \varphi = D(E(\varphi) \circ e_A) \circ \varepsilon_X.
\]
Duality

If $e_A : A \rightarrow ED(A)$ is surjective and therefore an isomorphism, for all $A$ in $\mathcal{A}$, then we say that $M$ yields a duality on $\mathcal{A}$ (or that $M$ dualises $M$).
Duality

If $e_A : A \rightarrow ED(A)$ is surjective and therefore an isomorphism, for all $A$ in $\mathcal{A}$, then we say that $\mathcal{M}$ yields a duality on $\mathcal{A}$ (or that $\mathcal{M}$ dualises $\mathcal{M}$).

Equivalently, $\mathcal{M}$ yields a duality on $\mathcal{A}$ if the dual adjunction $\langle D, E, e, \varepsilon \rangle$ is a dual category equivalence between $\mathcal{A}$ and a full subcategory of $\mathcal{X}$. 
If, in addition, $\varepsilon_X : X \rightarrow DE(X)$ is a surjection and therefore an isomorphism, for all $X$ in $\mathcal{X}$, then $\mathcal{M}$ yields a full duality on $\mathcal{A}$ (or $\mathcal{M}$ fully dualises $\mathcal{M}$).

Equivalently, $\mathcal{M}$ yields a full duality on $\mathcal{A}$ if the dual adjunction $\langle D, E, e, \varepsilon \rangle$ is a dual category equivalence between $\mathcal{A}$ and $\mathcal{X}$. 
Embeddings, injectivity and strong duality

Let $M$ be any alter ego of an algebra $M$, and let

$$D: \mathcal{A} \rightarrow \mathcal{X} \quad \text{and} \quad E: \mathcal{X} \rightarrow \mathcal{A}$$

be the induced hom-functors.

It is easy to see that:

- $D$ and $E$ send surjections to embeddings,
- $D$ sends embeddings in $\mathcal{A}$ to surjections in $\mathcal{X}$ if and only if $M$ is injective in $\mathcal{A}$, and
- $E$ sends embeddings in $\mathcal{X}$ to surjections in $\mathcal{A}$ if and only if $M$ is injective in $\mathcal{X}$.

Strong duality

If $M$ fully dualises $M$ and $M$ is injective in $\mathcal{X}$ (so that surjections in $\mathcal{A}$ correspond to embeddings in $\mathcal{X}$), we say that $M$ yields a strong duality on $\mathcal{A}$ (or that $M$ strongly dualises $M$).

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Further examples

- All three of our original examples — Stone duality, Priestley duality and Pontryagin duality — are examples of strong dualities.
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- Every finite lattice-based algebra admits a strong duality. [Davey, Werner 1980 and Clark, Davey 1995]

- The unary algebra admits a duality, but not a full duality. [Hyndman, Willard 2000]

- There is an example of a three-element algebra that admits a full duality that can not be upgraded to a strong duality. [Pitkethly 2009]

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Outline

Examples of natural dualities

Natural dualities: the basics

Duality theorems

- Duals of free algebras
- (IC) and the Second Duality Theorem
- Priestley duality via the Second Duality Theorem
- Further applications of the Second Duality Theorem
For duality, relations will do

- Let $\mathbf{M} = \langle M; F \rangle$ be a finite algebra,
- let $\mathbf{M} \sim = \langle M; G, H, R, T \rangle$ be an alter ego of $\mathbf{M}$, and
- define $\mathcal{A} := \text{ISP}(\mathbf{M})$ and $\mathcal{X} := \text{IScP}^+(\mathbf{M})$. 
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Recall that to prove that $\mathbf{M} \sim$ yields a duality on $\mathcal{A}$, it remains to show that

- for all $A \in \mathcal{A}$, the evaluation maps $e_A$, for $a \in A$, are the only $\mathcal{X}$-morphisms from $\mathcal{A}(A, \mathbf{M})$ to $\mathbf{M}$. 

Lemma (2.1.2) Let $\mathbf{M} \sim = \langle M; G, H, R, J \rangle$, define $\mathbf{M} \sim' = \langle M; R', T' \rangle$ where $R' := R \cup \{\text{graph}(h) | h \in G \cup H\}$.

Then $\mathbf{M} \sim$ yields a duality on $\mathcal{A}$ if and only if $\mathbf{M} \sim'$ does.

Thus, as far as obtaining a duality is concerned, we can restrict our attention to purely relational alter egos.
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Lemma (2.1.2)

Let $\mathcal{M} = \langle M; G, H, R, T \rangle$, define $\mathcal{M}' = \langle M; R', T \rangle$ where

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Thus, as far as obtaining a duality is concerned, we can restrict our attention to purely relational alter egos.
Given a non-empty set $S$, the set
\[ F_M(S) = \{ t : M^S \to M \mid t \text{ is an } S\text{-ary term function on } M \} \]
is the free $S$-generated algebra in $\mathcal{A}$ (the projections $\pi_s : M^S \to M$, for $s \in S$, are the free generators).
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In fact, we have $F_M(S) = \mathcal{X}(M^S, M)$. 

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The interpolation condition (IC)

Let $\mathcal{A}_{\text{fin}}$ and $\mathcal{X}_{\text{fin}}$ consist of the finite members of $\mathcal{A}$ and $\mathcal{X}$.

**Lemma (2.2.5)**

The following are equivalent:

(i) (IC) for each $n \in \mathbb{N}$ and each substructure $\mathbb{X}$ of $\mathcal{M}^n$, every morphism $\alpha : \mathbb{X} \to \mathcal{M}$ extends to a term function $t : \mathcal{M}^n \to \mathcal{M}$ of the algebra $\mathcal{M}$,

(ii) $(\text{INJ})^+_\text{fin}$ $\mathcal{M}$ is injective in $\mathcal{X}_{\text{fin}}$, and

$(\text{CLO})$ for each $n \in \mathbb{N}$, every morphism $t : \mathcal{M}^n \to \mathcal{M}$ is an $n$-ary term function on $\mathcal{M}$,

(iii) $\mathcal{M}$ yields a duality on $\mathcal{A}_{\text{fin}}$ and is injective in $\mathcal{X}_{\text{fin}}$. 
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We would like to obtain a duality for $\mathcal{A}$ in two steps:

- first show that $\mathcal{M}$ yields a duality on $\mathcal{A}_{\text{fin}}$, then...
The interpolation condition (IC)

Let $A_{\text{fin}}$ and $X_{\text{fin}}$ consist of the finite members of $A$ and $X$.

**Lemma (2.2.5)**

The following are equivalent:

(i) *(IC)* for each $n \in \mathbb{N}$ and each substructure $X$ of $M^n$, every morphism $\alpha: X \to M$ extends to a term function $t: M^n \to M$ of the algebra $M$,

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$(\text{CLO})$ for each $n \in \mathbb{N}$, every morphism $t: M^n \to M$ is an $n$-ary term function on $M$,

(iii) $M$ yields a duality on $A_{\text{fin}}$ and is injective in $X_{\text{fin}}$.

We would like to obtain a duality for $A$ in two steps:

- first show that $M$ yields a duality on $A_{\text{fin}}$, then
- apply some general theory to show that the duality lifts automatically to a duality on the whole of $A$. 

This is achievable provided $M$ enjoys some degree of finiteness.
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- apply some general theory to show that the duality lifts automatically to a duality on the whole of $\mathcal{A}$.

This is achievable provided $\mathcal{M}$ enjoys some degree of finiteness.
The Second Duality Theorem

If $\mathcal{M} = \langle M; G, R, \mathcal{T} \rangle$, that is, the type of $\mathcal{M}$ includes no partial operations, then we call $\mathcal{M}$ a total structure.

Theorem (2.2.7 Second Duality Theorem)
Assume that $\mathcal{M}$ is a total structure with $R$ finite. If (IC) holds, then $\mathcal{M}$ yields a duality on $\mathcal{A}$ and is injective in $\mathcal{X}$. 

This result is rather surprising. ▶ It gives us simple finitary conditions which yield both a dual adjunction between the categories $\mathcal{A}$ and $\mathcal{X}$ and a topological representation of every algebra in $\mathcal{A}$, but it requires us to do no category theory and no topology!
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Priestley duality via the Second Duality Theorem

Recall that

- $\mathbb{D} = \langle \{0, 1\}; \lor, \land, 0, 1 \rangle$ is the two-element bounded lattice,
- $\mathbb{D} \sim = \langle \{0, 1\}; \leq, T \rangle$ is the two-element chain endowed with the discrete topology.

**Theorem (Half of Priestley duality)**

$\mathbb{D} \sim$ yields a duality on the class $\mathcal{D} := \text{ISP}(\mathbb{D})$ of bounded distributive lattices, i.e., $e_A : A \to ED(A)$ is an isomorphism, for all $A \in \mathcal{D}$.

**Proof.**

We will prove that (IC) holds. Let $X$ be a substructure of $\mathcal{D}^n$ and let $\varphi : X \to \mathcal{D}$ be a morphism, i.e., $\varphi$ is order-preserving.

[We need to find a term function $t : \{0, 1\}^n \to \{0, 1\}$ on $\mathbb{D}$ such that $t(x) = \varphi(x)$, for all $x \in X$.]
The proof continued

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If $\varphi^{-1}(1) = \emptyset$, then define $t(v_1, \ldots, v_n) = 0$, and if $\varphi^{-1}(1) = X$, then define $t(v_1, \ldots, v_n) = 1$. 
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Otherwise, define $t(v_1, \ldots, v_n)$ by

$$t(v_1, \ldots, v_n) := \bigvee_{a \in \varphi^{-1}(1), a_i = 1} \left( \bigwedge_{a \in \varphi^{-1}(1)} v_i \right).$$
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If $t(x) = 1$, then there exists $a \in \varphi^{-1}(1)$ with $a_i = 1 \Rightarrow x_i = 1$.
Hence $\varphi(a) = 1$ and $a \leq x$. As $\varphi$ is order-preserving, we have $\varphi(x) = 1$. 

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If $t(x) = 1$, then there exists $a \in \varphi^{-1}(1)$ with $a_i = 1 \Rightarrow x_i = 1$.
Hence $\varphi(a) = 1$ and $a \leq x$. As $\varphi$ is order-preserving, we have $\varphi(x) = 1$. Hence $t(x) = \varphi(x)$, for all $x \in X$. □
Further applications of the Second Duality Theorem

Some exercises for you. In each case, prove that (IC) holds.

(1) [Stone] Let $\mathcal{B} = \langle \{0, 1\}; \lor, \land, ', 0, 1 \rangle$; then $\mathcal{B} = \text{ISP}(\mathcal{B})$ is the class of Boolean algebras. Show that $\mathcal{B} \cong \langle \{0, 1\}; \top \rangle$ yields a duality on $\mathcal{B}$.

(2) [Priestley] Let $\mathcal{L} = \langle \{0, 1\}; \lor, \land \rangle$; then $\mathcal{L} = \text{ISP}(\mathcal{L})$ is the class of distributive lattices. Show that $\mathcal{L} \cong \langle \{0, 1\}; 0, 1, \leq, \top \rangle$ yields a duality on $\mathcal{L}$.

(3) [Hofmann–Mislove–Stralka] Let $\mathcal{S} = \langle \{0, 1\}; \land \rangle$; then $\mathcal{S} = \text{ISP}(\mathcal{S})$ is the class of meet semilattices. Show that $\mathcal{S} \cong \langle \{0, 1\}; \land, 0, 1, \top \rangle$ yields a duality on $\mathcal{S}$.

(4) [Pontryagin] Let $\mathcal{Z}_m = \langle \mathbb{Z}_m; +, -, 0 \rangle$; then $\mathcal{A}_m = \text{ISP}(\mathcal{Z}_m)$ is the class of abelian groups of exponent $m$. Show that $\mathcal{Z} \cong \langle \mathbb{Z}_m; +, -, 0, \top \rangle$ yields a duality on $\mathcal{A}_m$. 