Lecture 1: an invitation to Priestley duality

Brian A. Davey

TACL 2015 School
Campus of Salerno (Fisciano)
15–19 June 2015

Outline

Outline

Bounded distributive lattices

Priestley duality for finite distributive lattices

Priestley duality via homsets

Priestley duality for infinite distributive lattices

Examples of Priestley spaces

The translation industry: restricted Priestley duals

Useful facts about Priestley spaces

Concrete examples of bounded distributive lattices

1. The two-element chain

   $\underline{\mathbf{2}} = \langle \{0, 1\}; \vee, \wedge, 0, 1 \rangle.$

2. All subsets of a set S:

   $\langle \wp(S); \cup, \cap, \varnothing, S \rangle.$

3. Finite or cofinite subsets of $\mathbb{N}$:

   $\langle \wp_{\mathrm{FC}}(\mathbb{N}); \cup, \cap, \varnothing, \mathbb{N} \rangle.$

4. Open subsets of a topological space $\mathbf{X}$:

   $\langle \mathcal{O}(\mathbf{X}); \cup, \cap, \varnothing, X \rangle.$

More examples of bounded distributive lattices

5. $\langle\{T, F\};$ or, and, F, T$\rangle$.

6. $\langle\mathbb{N}\cup\{0\};$ lcm, gcd, $1, 0\rangle$.

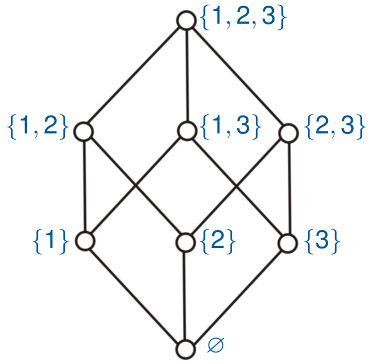
   (Use the fact that, $\mathrm{lcm}(m,n)\cdot\mathrm{gcd}(m,n) = mn$, for all $m, n \in \mathbb{N}\cup\{0\}$, and that a lattice is distributive iff it satisfies $x \vee z = y \vee z \;\;\&\;\; x \wedge z = y \wedge z \implies x = y$.)

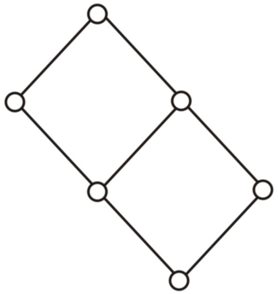
7. Subgroups of a cyclic group $\mathbf{G}$,

   $\langle \mathrm{Sub}(\mathbf{G}); \vee, \cap, \{e\}, G\rangle$, where $H \vee K := \mathrm{sg}_{\mathbf{G}}(H \cup K)$.

Drawing distributive lattices

Any distributive lattice $\langle L; \vee, \wedge, 0, 1\rangle$ has a natural order corresponding to set inclusion: $a \leqslant b \iff a \vee b = b$.

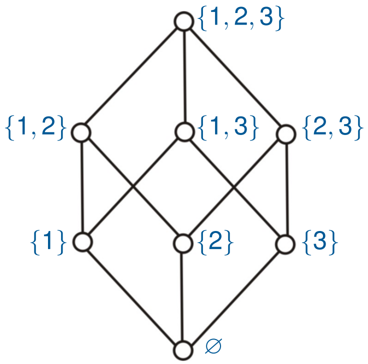


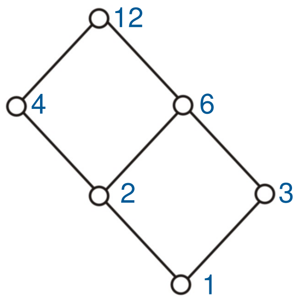


$\vee$ union
$\wedge$ intersection
$\leqslant$ inclusion

Drawing distributive lattices

Any distributive lattice $\langle L; \vee, \wedge, 0, 1\rangle$ has a natural order corresponding to set inclusion: $a \leqslant b \iff a \vee b = b$.



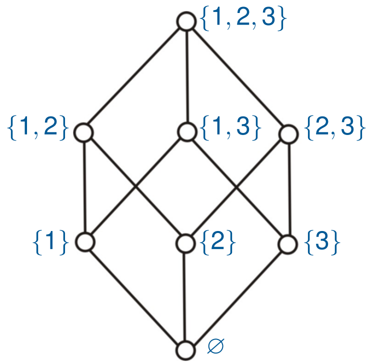


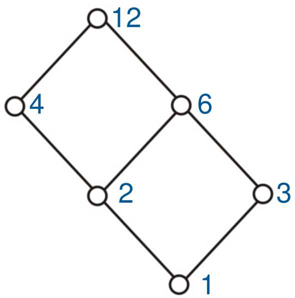
$\vee$ union
$\wedge$ intersection
$\leqslant$ inclusion

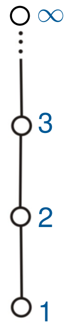
$\vee$ lcm
$\wedge$ gcd
$\leqslant$ division

Drawing distributive lattices

Any distributive lattice $\langle L; \vee, \wedge, 0, 1\rangle$ has a natural order corresponding to set inclusion: $a \leqslant b \iff a \vee b = b$.





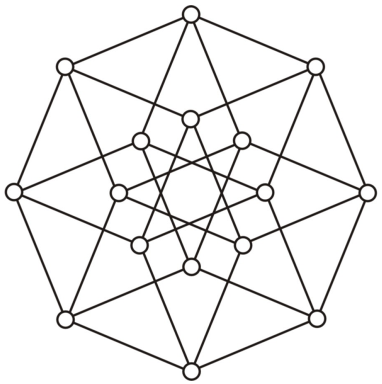


$\vee$ union
$\wedge$ intersection
$\leqslant$ inclusion

$\vee$ lcm
$\wedge$ gcd
$\leqslant$ division

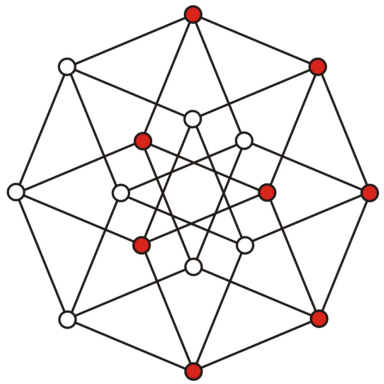
$\vee$ max
$\wedge$ min
$\leqslant$ usual

A sublattice $\mathbf{L}$ of $\underline{\mathbf{2}}^4$

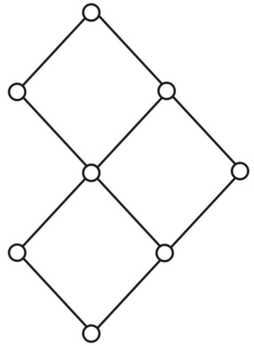


$\underline{\mathbf{2}}^4$

A sublattice $\mathbf{L}$ of $\underline{\mathbf{2}}^4$

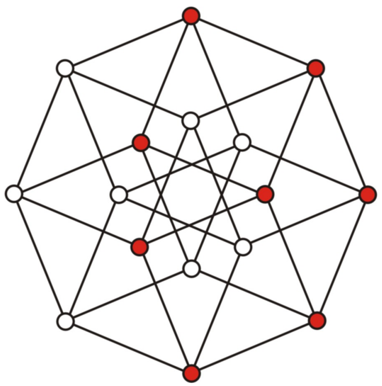


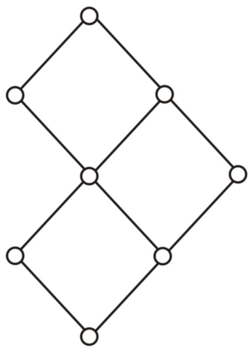
$\underline{\mathbf{2}}^4$



$\mathbf{L}$

A sublattice $\mathbf{L}$ of $\underline{\mathbf{2}}^4$





$\underline{\mathbf{2}}^4$

Note: Every distributive lattice embeds into $\underline{\mathbf{2}}^S$, for some set S.

Outline

Representing finite distributive lattices

Birkhoff's representation for a finite distributive lattice $\mathbf{L}$

Let $\mathbf{L}$ be a finite distributive lattice.

$\mathbf{L}$ is isomorphic to the collection $\mathcal{O}(\mathbf{P})$ of all down-sets of an ordered set $\mathbf{P} = \langle P; \leqslant \rangle$, under union, intersection, $\varnothing$ and P.

Representing finite distributive lattices

Birkhoff's representation for a finite distributive lattice $\mathbf{L}$

Let $\mathbf{L}$ be a finite distributive lattice.

$\mathbf{L}$ is isomorphic to the collection $\mathcal{O}(\mathbf{P})$ of all down-sets of an ordered set $\mathbf{P} = \langle P; \leqslant \rangle$, under union, intersection, $\varnothing$ and P.

In fact, we can choose $\mathbf{P}$ to be the ordered set $\langle \mathcal{J}(\mathbf{L}); \leqslant \rangle$ of join-irreducible elements of $\mathbf{L}$.

Representing finite distributive lattices

Birkhoff's representation for a finite distributive lattice $\mathbf{L}$

Let $\mathbf{L}$ be a finite distributive lattice.

$\mathbf{L}$ is isomorphic to the collection $\mathcal{O}(\mathbf{P})$ of all down-sets of an ordered set $\mathbf{P} = \langle P; \leqslant \rangle$, under union, intersection, $\varnothing$ and P.

In fact, we can choose $\mathbf{P}$ to be the ordered set $\langle \mathcal{J}(\mathbf{L}); \leqslant \rangle$ of join-irreducible elements of $\mathbf{L}$.

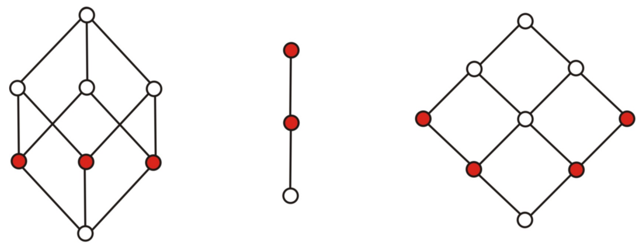
Theorem [G. Birkhoff]
Let $\mathbf{L}$ be a finite distributive lattice and let $\mathbf{P}$ be a finite ordered set. Then

- *$\mathbf{L}$ is isomorphic to $\mathcal{O}(\mathcal{J}(\mathbf{L}))$, and*
- *$\mathbf{P}$ is isomorphic to $\mathcal{J}(\mathcal{O}(\mathbf{P}))$.*

More examples

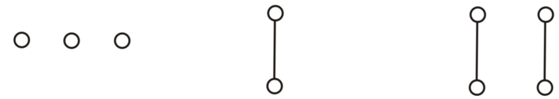
Distributive lattice

$\mathbf{L} \cong \mathcal{O}(\mathcal{J}(\mathbf{L}))$



Ordered set

$\mathbf{P} \cong \mathcal{J}(\mathcal{O}(\mathbf{P}))$

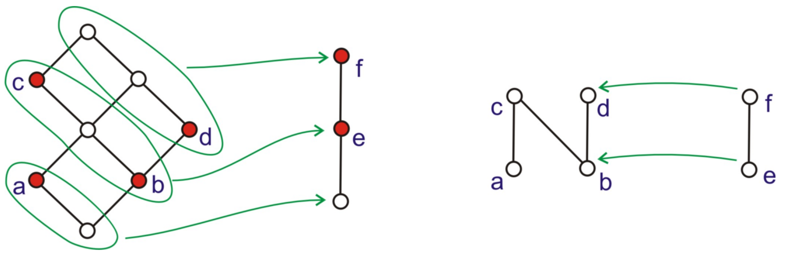


Duality for finite distributive lattices

The classes of

finite distributive lattices and finite ordered sets

are dually equivalent.

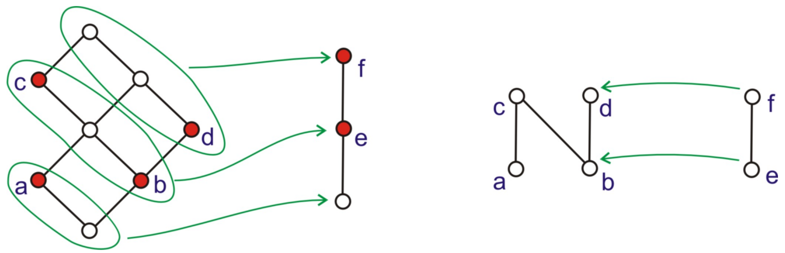


Duality for finite distributive lattices

The classes of

finite distributive lattices and finite ordered sets

are dually equivalent.



surjections	$\longleftrightarrow$	embeddings
embeddings	$\longleftrightarrow$	surjections
products	$\longleftrightarrow$	disjoint unions

Outline

Duals of finite bounded distributive lattices

Let $\mathbf{L} = \langle L; \vee, \wedge, 0, 1\rangle$ be a finite bounded distributive lattice.
We can define its dual $D(\mathbf{L})$ to be either

- $\mathcal{J}(\mathbf{L})$ — the ordered set of join-irreducible elements of $\mathbf{L}$

or

- $\mathcal{D}(\mathbf{L}, \underline{\mathbf{2}})$ — the ordered set of $\{0, 1\}$-homomorphisms from $\mathbf{L}$ to the two-element bounded lattice $\underline{\mathbf{2}} = \langle\{0, 1\}; \vee, \wedge, 0, 1\rangle$.

Here $\mathcal{D}$ denotes the category of bounded distributive lattices.

Duals of finite bounded distributive lattices

Let $\mathbf{L} = \langle L; \vee, \wedge, 0, 1 \rangle$ be a finite bounded distributive lattice.

We can define its dual $D(\mathbf{L})$ to be either

- $\mathcal{J}(\mathbf{L})$ — the ordered set of join-irreducible elements of $\mathbf{L}$

or

- $\mathcal{D}(\mathbf{L}, \underline{\mathbf{2}})$ — the ordered set of $\{0, 1\}$-homomorphisms from $\mathbf{L}$ to the two-element bounded lattice $\underline{\mathbf{2}} = \langle \{0, 1\}; \vee, \wedge, 0, 1 \rangle$.

In fact, we have the following dual order-isomorphism:

$$\mathcal{J}(\mathbf{L}) \cong^{\partial} \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}).$$

Here $\mathcal{D}$ denotes the category of bounded distributive lattices.

Duals of finite ordered sets

Let $\mathbf{P} = \langle P; \leqslant \rangle$ be a finite ordered set.

We can define its dual $E(\mathbf{P})$ to be either

- $\mathcal{O}(\mathbf{P})$ – the lattice of down-sets (= order ideals) of $\mathbf{P}$

or

- $\mathcal{P}(\mathbf{P}, \underset{\sim}{\mathbf{2}})$ – the lattice of order-preserving maps from $\mathbf{P}$ to the two-element ordered set $\underset{\sim}{\mathbf{2}} = \langle \{0, 1\}; \leqslant \rangle$.

Here $\mathcal{P}$ denotes the category of ordered sets.

(Warning! The definitions of $\underset{\sim}{\mathbf{2}}$ and $\mathcal{P}$ will change once we consider the infinite case.)

Duals of finite ordered sets

Let $\mathbf{P} = \langle P; \leqslant \rangle$ be a finite ordered set.

We can define its dual $E(\mathbf{P})$ to be either

- $\mathcal{O}(\mathbf{P})$ – the lattice of down-sets (= order ideals) of $\mathbf{P}$

or

- $\mathcal{P}(\mathbf{P}, \underset{\sim}{\mathbf{2}})$ – the lattice of order-preserving maps from $\mathbf{P}$ to the two-element ordered set $\underset{\sim}{\mathbf{2}} = \langle \{0, 1\}; \leqslant \rangle$.

In fact, we have the following dual lattice-isomorphism:

$$\mathcal{O}(\mathbf{P}) \cong^{\partial} \mathcal{P}(\mathbf{P}, \underset{\sim}{\mathbf{2}}).$$

Here $\mathcal{P}$ denotes the category of ordered sets.

(Warning! The definitions of $\underset{\sim}{\mathbf{2}}$ and $\mathcal{P}$ will change once we consider the infinite case.)

Duals of morphisms

Let $\mathbf{L}$ and $\mathbf{K}$ be a finite distributive lattices and let $\mathbf{P}$ and $\mathbf{Q}$ be a finite ordered sets.

- There is a bijection between the $\{0, 1\}$-homomorphisms from $\mathbf{L}$ to $\mathbf{K}$ and the order-preserving maps from $D(\mathbf{K})$ to $D(\mathbf{L})$. Given $f\colon \mathbf{L} \to \mathbf{K}$, we define

$$\varphi\colon \mathcal{J}(\mathbf{K}) \to \mathcal{J}(\mathbf{L}) \text{ by } \varphi(x) := \min(f^{-1}({\uparrow}x)),$$

$$\varphi\colon \mathcal{D}(\mathbf{K}, \underline{\mathbf{2}}) \to \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \text{ by } \varphi(x) := x \circ f.$$

We denote the map φ by $D(f)$.

Duals of morphisms

Let **L** and **K** be a finite distributive lattices and let **P** and **Q** be a finite ordered sets.

- There is a bijection between the $\{0,1\}$-homomorphisms from **L** to **K** and the order-preserving maps from $D(\mathbf{K})$ to $D(\mathbf{L})$. Given $f\colon \mathbf{L} \to \mathbf{K}$, we define

  $$\varphi\colon \mathcal{J}(\mathbf{K}) \to \mathcal{J}(\mathbf{L}) \text{ by } \varphi(x) := \min(f^{-1}(\uparrow x)),$$
  $$\varphi\colon \mathcal{D}(\mathbf{K}, \underline{\mathbf{2}}) \to \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \text{ by } \varphi(x) := x \circ f.$$

  We denote the map φ by $D(f)$.
- There is a bijection between the order-preserving maps from **P** to **Q** and the $\{0,1\}$-homomorphisms from $E(\mathbf{Q})$ to $E(\mathbf{P})$. Given $\varphi\colon \mathbf{P} \to \mathbf{Q}$, we define

  $$f\colon \mathcal{O}(\mathbf{Q}) \to \mathcal{O}(\mathbf{P}) \text{ by } f(A) := \varphi^{-1}(A),$$
  $$f\colon \mathcal{P}(\mathbf{Q}, \underset{\sim}{\mathbf{2}}) \to \mathcal{P}(\mathbf{P}, \underset{\sim}{\mathbf{2}}) \text{ by } f(\alpha) := \alpha \circ \varphi.$$

  We denote the map f by $E(\varphi)$.

The duality at the finite level

- $\underline{\mathbf{2}} = \langle \{0,1\}; \vee, \wedge, 0, 1 \rangle$ is the two-element lattice,
- $\underset{\sim}{\mathbf{2}} = \langle \{0,1\}; \leqslant \rangle$ is the two-element ordered set with $0 \leqslant 1$.

Define either

$$D(\mathbf{L}) := \mathcal{J}(\mathbf{L}) \quad \text{and} \quad E(\mathbf{P}) := \mathcal{O}(\mathbf{P})$$

or

$$D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \leqslant \underset{\sim}{\mathbf{2}}^{L} \quad \text{and} \quad E(\mathbf{P}) := \mathcal{P}(\mathbf{P}, \underset{\sim}{\mathbf{2}}) \leqslant \underline{\mathbf{2}}^{P}.$$

The duality at the finite level

- $\underline{\mathbf{2}} = \langle \{0,1\}; \vee, \wedge, 0, 1 \rangle$ is the two-element lattice,
- $\underset{\sim}{\mathbf{2}} = \langle \{0,1\}; \leqslant \rangle$ is the two-element ordered set with $0 \leqslant 1$.

Define either

$$D(\mathbf{L}) := \mathcal{J}(\mathbf{L}) \quad \text{and} \quad E(\mathbf{P}) := \mathcal{O}(\mathbf{P})$$

or

$$D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \leqslant \underset{\sim}{\mathbf{2}}^{L} \quad \text{and} \quad E(\mathbf{P}) := \mathcal{P}(\mathbf{P}, \underset{\sim}{\mathbf{2}}) \leqslant \underline{\mathbf{2}}^{P}.$$

Theorem [G. Birkhoff, H. A. Priestley]
Every finite distributive lattice is encoded by an ordered set:

$$\mathbf{L} \cong ED(\mathbf{L}) \quad \textit{and} \quad \mathbf{P} \cong DE(\mathbf{P}),$$

for each finite distributive lattice **L** *and finite ordered set* **P**.

Indeed, the categories of finite bounded distributive lattices and finite ordered sets are dually equivalent.

Outline

Infinite distributive lattices

Example

The finite-cofinite lattice $\mathbf{L} = \langle \wp_{\mathrm{FC}}(\mathbb{N}); \cup, \cap, \varnothing, \mathbb{N} \rangle$ cannot be obtained as the down-sets of an ordered set.

Infinite distributive lattices

Example

The finite-cofinite lattice $\mathbf{L} = \langle \wp_{\mathrm{FC}}(\mathbb{N}); \cup, \cap, \varnothing, \mathbb{N} \rangle$ cannot be obtained as the down-sets of an ordered set.

Proof.

- Since **L** is complemented, the ordered set would have to be an anti-chain.

Infinite distributive lattices

Example

The finite-cofinite lattice $\mathbf{L} = \langle \wp_{\mathrm{FC}}(\mathbb{N}); \cup, \cap, \varnothing, \mathbb{N} \rangle$ cannot be obtained as the down-sets of an ordered set.

Proof.

- Since **L** is complemented, the ordered set would have to be an anti-chain.
- Since **L** is infinite, the ordered set would have to be infinite.

Infinite distributive lattices

Example

The finite-cofinite lattice $\mathbf{L} = \langle \wp_{\mathrm{FC}}(\mathbb{N}); \cup, \cap, \varnothing, \mathbb{N} \rangle$ cannot be obtained as the down-sets of an ordered set.

Proof.

- Since **L** is complemented, the ordered set would have to be an anti-chain.
- Since **L** is infinite, the ordered set would have to be infinite.
- So there would be at least $2^{\mathbb{N}}$ down-sets.

Infinite distributive lattices

Example

The finite-cofinite lattice $\mathbf{L} = \langle \wp_{\mathrm{FC}}(\mathbb{N}); \cup, \cap, \varnothing, \mathbb{N} \rangle$ cannot be obtained as the down-sets of an ordered set.

Proof.

- Since $\mathbf{L}$ is complemented, the ordered set would have to be an anti-chain.
- Since $\mathbf{L}$ is infinite, the ordered set would have to be infinite.
- So there would be at least $2^{\mathbb{N}}$ down-sets.
- But $\wp_{\mathrm{FC}}(\mathbb{N})$ is countable. $\square$

Infinite distributive lattices

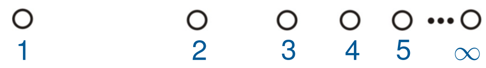
Example

The finite-cofinite lattice $\mathbf{L} = \langle \wp_{\mathrm{FC}}(\mathbb{N}); \cup, \cap, \varnothing, \mathbb{N} \rangle$ cannot be obtained as the down-sets of an ordered set.

Proof.

- Since $\mathbf{L}$ is complemented, the ordered set would have to be an anti-chain.
- Since $\mathbf{L}$ is infinite, the ordered set would have to be infinite.
- So there would be at least $2^{\mathbb{N}}$ down-sets.
- But $\wp_{\mathrm{FC}}(\mathbb{N})$ is countable. $\square$

But it can be obtained as the clopen down-sets of a topological ordered set.



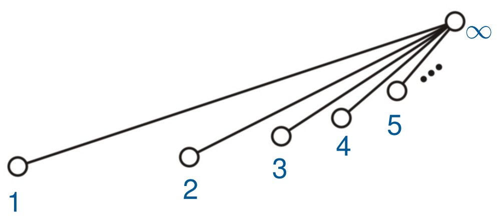
More examples

Distributive lattice:

All finite subsets of $\mathbb{N}$, as well as $\mathbb{N}$ itself,

$\langle \wp_{\mathrm{fin}}(\mathbb{N}) \cup \{\mathbb{N}\}; \cup, \cap, \varnothing, \mathbb{N} \rangle$.

Topological ordered set:



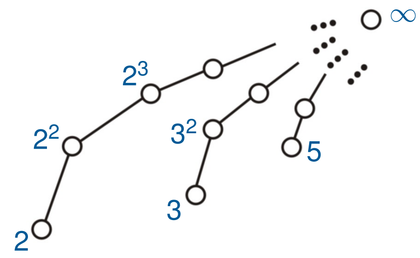
More examples

Distributive lattice:

All finite subsets of $\mathbb{N}$, as well as $\mathbb{N}$ itself,

$\langle \mathbb{N} \cup \{0\}; \mathrm{lcm}, \gcd, 1, 0 \rangle$.

Topological ordered set:



The duality in general

- $\underline{\mathbf{2}} = \langle \{0,1\}; \vee, \wedge, 0, 1 \rangle$ is the two-element bounded lattice.

In general, we need to endow the dual $D(\mathbf{L})$ of a bounded distributive lattice $\mathbf{L}$ with a topology. This is easy if we define the dual of $\mathbf{L}$ to be $D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}})$.

The duality in general

- $\underline{\mathbf{2}} = \langle \{0,1\}; \vee, \wedge, 0, 1 \rangle$ is the two-element bounded lattice.

In general, we need to endow the dual $D(\mathbf{L})$ of a bounded distributive lattice $\mathbf{L}$ with a topology. This is easy if we define the dual of $\mathbf{L}$ to be $D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}})$.

We first endow $\langle \{0,1\}; \leqslant \rangle$ with the discrete topology:

The duality in general

- $\underline{\mathbf{2}} = \langle \{0,1\}; \vee, \wedge, 0, 1 \rangle$ is the two-element bounded lattice.

In general, we need to endow the dual $D(\mathbf{L})$ of a bounded distributive lattice $\mathbf{L}$ with a topology. This is easy if we define the dual of $\mathbf{L}$ to be $D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}})$.

We first endow $\langle \{0,1\}; \leqslant \rangle$ with the discrete topology:

- $\underset{\sim}{\mathbf{2}} = \langle \{0,1\}; \leqslant, \mathcal{T} \rangle$ is the two-element ordered set with $0 \leqslant 1$ endowed with the discrete topology $\mathcal{T}$.

The duality in general

- $\underline{\mathbf{2}} = \langle \{0,1\}; \vee, \wedge, 0, 1 \rangle$ is the two-element bounded lattice.

In general, we need to endow the dual $D(\mathbf{L})$ of a bounded distributive lattice $\mathbf{L}$ with a topology. This is easy if we define the dual of $\mathbf{L}$ to be $D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}})$.

We first endow $\langle \{0,1\}; \leqslant \rangle$ with the discrete topology:

- $\underset{\sim}{\mathbf{2}} = \langle \{0,1\}; \leqslant, \mathcal{T} \rangle$ is the two-element ordered set with $0 \leqslant 1$ endowed with the discrete topology $\mathcal{T}$.

Then we define

- $D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \leqslant \underset{\sim}{\mathbf{2}}^{L}$.

The duality in general

- $\underline{\mathbf{2}} = \langle \{0,1\}; \vee, \wedge, 0, 1 \rangle$ is the two-element bounded lattice.

In general, we need to endow the dual $D(\mathbf{L})$ of a bounded distributive lattice $\mathbf{L}$ with a topology. This is easy if we define the dual of $\mathbf{L}$ to be $D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}})$.

We first endow $\langle \{0,1\}; \leqslant \rangle$ with the discrete topology:

- $\underset{\sim}{\mathbf{2}} = \langle \{0,1\}; \leqslant, \mathcal{T} \rangle$ is the two-element ordered set with $0 \leqslant 1$ endowed with the discrete topology $\mathcal{T}$.

Then we define

- $D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \leqslant \underset{\sim}{\mathbf{2}}^L$.

We put the pointwise order and the product topology on $\underset{\sim}{\mathbf{2}}^L$.

The duality in general

- $\underline{\mathbf{2}} = \langle \{0,1\}; \vee, \wedge, 0, 1 \rangle$ is the two-element bounded lattice.

In general, we need to endow the dual $D(\mathbf{L})$ of a bounded distributive lattice $\mathbf{L}$ with a topology. This is easy if we define the dual of $\mathbf{L}$ to be $D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}})$.

We first endow $\langle \{0,1\}; \leqslant \rangle$ with the discrete topology:

- $\underset{\sim}{\mathbf{2}} = \langle \{0,1\}; \leqslant, \mathcal{T} \rangle$ is the two-element ordered set with $0 \leqslant 1$ endowed with the discrete topology $\mathcal{T}$.

Then we define

- $D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \leqslant \underset{\sim}{\mathbf{2}}^L$.

We put the pointwise order and the product topology on $\underset{\sim}{\mathbf{2}}^L$.

Then $D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}})$ inherits its order and topology from $\underset{\sim}{\mathbf{2}}^L$.

The duality in general

- $\underline{\mathbf{2}} = \langle \{0,1\}; \vee, \wedge, 0, 1 \rangle$ is the two-element bounded lattice.

In general, we need to endow the dual $D(\mathbf{L})$ of a bounded distributive lattice $\mathbf{L}$ with a topology. This is easy if we define the dual of $\mathbf{L}$ to be $D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}})$.

We first endow $\langle \{0,1\}; \leqslant \rangle$ with the discrete topology:

- $\underset{\sim}{\mathbf{2}} = \langle \{0,1\}; \leqslant, \mathcal{T} \rangle$ is the two-element ordered set with $0 \leqslant 1$ endowed with the discrete topology $\mathcal{T}$.

Then we define

- $D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \leqslant \underset{\sim}{\mathbf{2}}^L$.

We put the pointwise order and the product topology on $\underset{\sim}{\mathbf{2}}^L$.

Then $D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}})$ inherits its order and topology from $\underset{\sim}{\mathbf{2}}^L$.

$\mathcal{D}(\mathbf{L}, \underline{\mathbf{2}})$ is a topologically closed subset of $\underset{\sim}{\mathbf{2}}^L$ (easy exercise).

The duality in general

- $\underline{\mathbf{2}} = \langle \{0,1\}; \vee, \wedge, 0, 1 \rangle$ is the two-element bounded lattice.

In general, we need to endow the dual $D(\mathbf{L})$ of a bounded distributive lattice $\mathbf{L}$ with a topology. This is easy if we define the dual of $\mathbf{L}$ to be $D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}})$.

We first endow $\langle \{0,1\}; \leqslant \rangle$ with the discrete topology:

- $\underset{\sim}{\mathbf{2}} = \langle \{0,1\}; \leqslant, \mathcal{T} \rangle$ is the two-element ordered set with $0 \leqslant 1$ endowed with the discrete topology $\mathcal{T}$.

Then we define

- $D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \leqslant \underset{\sim}{\mathbf{2}}^L$.

We put the pointwise order and the product topology on $\underset{\sim}{\mathbf{2}}^L$.

Then $D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}})$ inherits its order and topology from $\underset{\sim}{\mathbf{2}}^L$.

$\mathcal{D}(\mathbf{L}, \underline{\mathbf{2}})$ is a topologically closed subset of $\underset{\sim}{\mathbf{2}}^L$ (easy exercise).

Hence $D(\mathbf{L})$ is a compact ordered topological space.

Priestley spaces

The ordered space $D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \leqslant \underset{\sim}{\mathbf{2}}^L$ is more than a compact ordered space. It is a Priestley space.

A topological structure $\mathbf{X} = \langle X; \leqslant, \mathcal{T}\rangle$ is a Priestley space if

- $\langle X; \leqslant, \rangle$ is an ordered set,
- $\mathcal{T}$ is a compact topology on X, and
- for all $x, y \in X$ with $x \nleqslant y$, there is a clopen down-set A of $\mathbf{X}$ such that $x \notin A$ and $y \in A$.

The category of Priestley spaces is denoted by $\mathcal{P}$.

Priestley spaces

The ordered space $D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \leqslant \underset{\sim}{\mathbf{2}}^L$ is more than a compact ordered space. It is a Priestley space.

A topological structure $\mathbf{X} = \langle X; \leqslant, \mathcal{T}\rangle$ is a Priestley space if

- $\langle X; \leqslant, \rangle$ is an ordered set,
- $\mathcal{T}$ is a compact topology on X, and
- for all $x, y \in X$ with $x \nleqslant y$, there is a clopen down-set A of $\mathbf{X}$ such that $x \notin A$ and $y \in A$.

The category of Priestley spaces is denoted by $\mathcal{P}$.
The following result is very easy to prove.

Lemma

- *$D(\mathbf{L}) = \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \leqslant \underset{\sim}{\mathbf{2}}^L$ is a Priestley space, for every bounded distributive lattice.*
- *$E(\mathbf{X}) = \mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{2}}) \leqslant \underline{\mathbf{2}}^X$ is a bounded distributive lattice, for every Priestley space $\mathbf{X}$.*

The functors

We now have functors $D\colon \mathcal{D} \to \mathcal{P}$ and $E\colon \mathcal{P} \to \mathcal{D}$ given by

$$D(\mathbf{L}) = \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \leqslant \underset{\sim}{\mathbf{2}}^L \quad \text{and} \quad E(\mathbf{X}) = \mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{2}}) \leqslant \underline{\mathbf{2}}^X.$$

The functors

We now have functors $D\colon \mathcal{D} \to \mathcal{P}$ and $E\colon \mathcal{P} \to \mathcal{D}$ given by

$$D(\mathbf{L}) = \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \leqslant \underset{\sim}{\mathbf{2}}^L \quad \text{and} \quad E(\mathbf{X}) = \mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{2}}) \leqslant \underline{\mathbf{2}}^X.$$

D and E are defined on morphisms via composition exactly as they were in the finite case:

- given $f\colon \mathbf{L} \to \mathbf{K}$, we define
$$D(f)\colon \mathcal{D}(\mathbf{K}, \underline{\mathbf{2}}) \to \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \ \text{ by } D(f)(x) := x \circ f;$$
- given $\varphi\colon \mathbf{X} \to \mathbf{Y}$, we define
$$E(\varphi)\colon \mathcal{P}(\mathbf{Y}, \underset{\sim}{\mathbf{2}}) \to \mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{2}}) \ \text{ by } E(\varphi)(\alpha) := \alpha \circ \varphi.$$

The functors

We now have functors $D\colon \mathcal{D} \to \mathcal{P}$ and $E\colon \mathcal{P} \to \mathcal{D}$ given by

$$D(\mathbf{L}) = \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \leqslant \underset{\sim}{\mathbf{2}}^{L} \quad \text{and} \quad E(\mathbf{X}) = \mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{2}}) \leqslant \underline{\mathbf{2}}^{X}.$$

D and E are defined on morphisms via composition exactly as they were in the finite case:

- given $f\colon \mathbf{L} \to \mathbf{K}$, we define

$$D(f)\colon \mathcal{D}(\mathbf{K}, \underline{\mathbf{2}}) \to \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \text{ by } D(f)(x) := x \circ f;$$

- given $\varphi\colon \mathbf{X} \to \mathbf{Y}$, we define

$$E(\varphi)\colon \mathcal{P}(\mathbf{Y}, \underset{\sim}{\mathbf{2}}) \to \mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{2}}) \text{ by } E(\varphi)(\alpha) := \alpha \circ \varphi.$$

The functors are contravariant as they reverse the direction of the morphisms: $D(f)\colon D(\mathbf{K}) \to D(\mathbf{L})$ and $E(\varphi)\colon E(\mathbf{Y}) \to E(\mathbf{X})$.

The functors

We now have functors $D\colon \mathcal{D} \to \mathcal{P}$ and $E\colon \mathcal{P} \to \mathcal{D}$ given by

$$D(\mathbf{L}) = \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \leqslant \underset{\sim}{\mathbf{2}}^{L} \quad \text{and} \quad E(\mathbf{X}) = \mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{2}}) \leqslant \underline{\mathbf{2}}^{X}.$$

D and E are defined on morphisms via composition exactly as they were in the finite case:

- given $f\colon \mathbf{L} \to \mathbf{K}$, we define

$$D(f)\colon \mathcal{D}(\mathbf{K}, \underline{\mathbf{2}}) \to \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \text{ by } D(f)(x) := x \circ f;$$

- given $\varphi\colon \mathbf{X} \to \mathbf{Y}$, we define

$$E(\varphi)\colon \mathcal{P}(\mathbf{Y}, \underset{\sim}{\mathbf{2}}) \to \mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{2}}) \text{ by } E(\varphi)(\alpha) := \alpha \circ \varphi.$$

The functors are contravariant as they reverse the direction of the morphisms: $D(f)\colon D(\mathbf{K}) \to D(\mathbf{L})$ and $E(\varphi)\colon E(\mathbf{Y}) \to E(\mathbf{X})$.

23 / 35

The functors

We now have functors $D\colon \mathcal{D} \to \mathcal{P}$ and $E\colon \mathcal{P} \to \mathcal{D}$ given by

$$D(\mathbf{L}) = \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \leqslant \underset{\sim}{\mathbf{2}}^{L} \quad \text{and} \quad E(\mathbf{X}) = \mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{2}}) \leqslant \underline{\mathbf{2}}^{X}.$$

D and E are defined on morphisms via composition exactly as they were in the finite case:

- given $f\colon \mathbf{L} \to \mathbf{K}$, we define

$$D(f)\colon \mathcal{D}(\mathbf{K}, \underline{\mathbf{2}}) \to \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \text{ by } D(f)(x) := x \circ f;$$

- given $\varphi\colon \mathbf{X} \to \mathbf{Y}$, we define

$$E(\varphi)\colon \mathcal{P}(\mathbf{Y}, \underset{\sim}{\mathbf{2}}) \to \mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{2}}) \text{ by } E(\varphi)(\alpha) := \alpha \circ \varphi.$$

The functors are contravariant as they reverse the direction of the morphisms: $D(f)\colon D(\mathbf{K}) \to D(\mathbf{L})$ and $E(\varphi)\colon E(\mathbf{Y}) \to E(\mathbf{X})$.

23 / 35

The natural transformations

Let $\mathbf{L} \in \mathcal{D}$ and let $\mathbf{X} \in \mathcal{P}$. There are natural maps

$$e_{\mathbf{L}}\colon \mathbf{L} \to ED(\mathbf{L}) \quad \text{and} \quad \varepsilon_{\mathbf{X}}\colon \mathbf{X} \to DE(\mathbf{X})$$

to the double duals;

The natural transformations

Let $\mathbf{L} \in \mathcal{D}$ and let $\mathbf{X} \in \mathcal{P}$. There are natural maps

$$e_{\mathbf{L}} : \mathbf{L} \to ED(\mathbf{L}) \quad \text{and} \quad \varepsilon_{\mathbf{X}} : \mathbf{X} \to DE(\mathbf{X})$$

to the double duals;

The natural transformations

Let $\mathbf{L} \in \mathcal{D}$ and let $\mathbf{X} \in \mathcal{P}$. There are natural maps

$$e_{\mathbf{L}} : \mathbf{L} \to ED(\mathbf{L}) \quad \text{and} \quad \varepsilon_{\mathbf{X}} : \mathbf{X} \to DE(\mathbf{X})$$

to the double duals; namely,

- $e_{\mathbf{L}} : \mathbf{L} \to \mathcal{P}(\mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}), \underset{\sim}{\mathbf{2}})$ given by $a \mapsto e_{\mathbf{L}}(a)$,
  where $e_{\mathbf{L}}(a) : \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \to \underset{\sim}{\mathbf{2}} \ : x \mapsto x(a)$,

The natural transformations

Let $\mathbf{L} \in \mathcal{D}$ and let $\mathbf{X} \in \mathcal{P}$. There are natural maps

$$e_{\mathbf{L}} : \mathbf{L} \to ED(\mathbf{L}) \quad \text{and} \quad \varepsilon_{\mathbf{X}} : \mathbf{X} \to DE(\mathbf{X})$$

to the double duals; namely,

- $e_{\mathbf{L}} : \mathbf{L} \to \mathcal{P}(\mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}), \underset{\sim}{\mathbf{2}})$ given by $a \mapsto e_{\mathbf{L}}(a)$,
  where $e_{\mathbf{L}}(a) : \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \to \underset{\sim}{\mathbf{2}} \ : x \mapsto x(a)$,
- $\varepsilon_{\mathbf{X}} : \mathbf{X} \to \mathcal{D}(\mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{2}}), \underline{\mathbf{2}})$ given by $x \mapsto \varepsilon_{\mathbf{X}}(x)$,
  where $\varepsilon_{\mathbf{X}}(x) : \mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{2}}) \to \underline{\mathbf{2}} \ : \alpha \mapsto \alpha(x)$.

The natural transformations

Let $\mathbf{L} \in \mathcal{D}$ and let $\mathbf{X} \in \mathcal{P}$. There are natural maps

$$e_{\mathbf{L}} : \mathbf{L} \to ED(\mathbf{L}) \quad \text{and} \quad \varepsilon_{\mathbf{X}} : \mathbf{X} \to DE(\mathbf{X})$$

to the double duals; namely,

- $e_{\mathbf{L}} : \mathbf{L} \to \mathcal{P}(\mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}), \underset{\sim}{\mathbf{2}})$ given by $a \mapsto e_{\mathbf{L}}(a)$,
  where $e_{\mathbf{L}}(a) : \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \to \underset{\sim}{\mathbf{2}} \ : x \mapsto x(a)$,
- $\varepsilon_{\mathbf{X}} : \mathbf{X} \to \mathcal{D}(\mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{2}}), \underline{\mathbf{2}})$ given by $x \mapsto \varepsilon_{\mathbf{X}}(x)$,
  where $\varepsilon_{\mathbf{X}}(x) : \mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{2}}) \to \underline{\mathbf{2}} \ : \alpha \mapsto \alpha(x)$.

Priestley duality tells us that these maps are isomorphisms
(in $\mathcal{D}$ or $\mathcal{P}$, as appropriate).

Priestley duality

Theorem (Priestley duality)

- *The functors $D\colon \mathcal{D} \to \mathcal{P}$ and $E\colon \mathcal{P} \to \mathcal{D}$ give a dual category equivalence between $\mathcal{D}$ and $\mathcal{P}$.*
- *In particular, $e_{\mathbf{L}}\colon \mathbf{L} \to ED(\mathbf{L})$ and $\varepsilon_{\mathbf{X}}\colon \mathbf{X} \to DE(\mathbf{X})$ are isomorphisms for all $\mathbf{L} \in \mathcal{D}$ and $\mathbf{X} \in \mathcal{P}$.*

Outline

Some ordered spaces

The following figure comes from Chapter 11 of Davey and Priestley: Introduction to Lattices and Order.

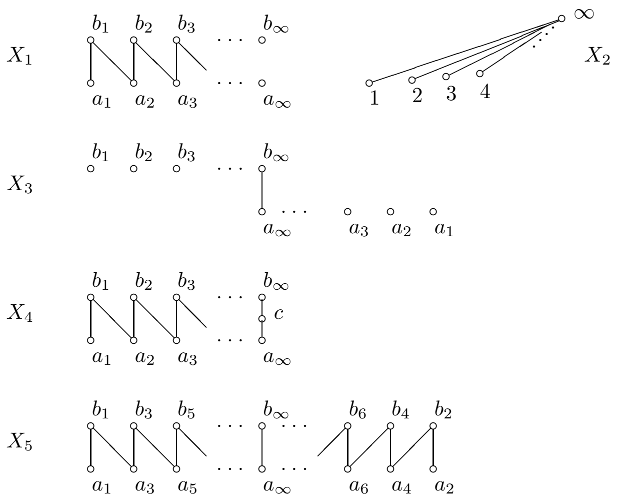


Figure 11.5

A subtlety

- If $\mathbf{X} = \langle X; \leqslant, \mathcal{T}\rangle$ is a Priestley space, then $\leqslant$ is a topologically closed subset of $X \times X$.

A subtlety

- If $\mathbf{X} = \langle X; \leqslant, \mathcal{T}\rangle$ is a Priestley space, then $\leqslant$ is a topologically closed subset of $X \times X$.
- [A. Stralka] Let $\mathbb{C} = \langle C; \leqslant, \mathcal{T}\rangle$, where

A subtlety

- If $\mathbf{X} = \langle X; \leqslant, \mathcal{T}\rangle$ is a Priestley space, then $\leqslant$ is a topologically closed subset of $X \times X$.
- [A. Stralka] Let $\mathbb{C} = \langle C; \leqslant, \mathcal{T}\rangle$, where
  - $\langle C; \mathcal{T}\rangle$ is the Cantor space created by successively deleting middle thirds of the unit interval, and

A subtlety

- If $\mathbf{X} = \langle X; \leqslant, \mathcal{T}\rangle$ is a Priestley space, then $\leqslant$ is a topologically closed subset of $X \times X$.
- [A. Stralka] Let $\mathbb{C} = \langle C; \leqslant, \mathcal{T}\rangle$, where
  - $\langle C; \mathcal{T}\rangle$ is the Cantor space created by successively deleting middle thirds of the unit interval, and
  - $x < y$ if and only if x and y are the endpoints of a deleted middle third.

A subtlety

- If $\mathbf{X} = \langle X; \leqslant, \mathcal{T}\rangle$ is a Priestley space, then $\leqslant$ is a topologically closed subset of $X \times X$.
- [A. Stralka] Let $\mathbb{C} = \langle C; \leqslant, \mathcal{T}\rangle$, where
  - $\langle C; \mathcal{T}\rangle$ is the Cantor space created by successively deleting middle thirds of the unit interval, and
  - $x < y$ if and only if x and y are the endpoints of a deleted middle third.

Then $\leqslant$ is closed in $C \times C$, but $\mathbb{C} = \langle C; \leqslant, \mathcal{T}\rangle$ is not a Priestley space.

A subtlety

- If $\mathbf{X} = \langle X; \leqslant, \mathcal{T}\rangle$ is a Priestley space, then $\leqslant$ is a topologically closed subset of $X \times X$.
- [A. Stralka] Let $\mathbb{C} = \langle C; \leqslant, \mathcal{T}\rangle$, where
  - $\langle C; \mathcal{T}\rangle$ is the Cantor space created by successively deleting middle thirds of the unit interval, and
  - $x < y$ if and only if x and y are the endpoints of a deleted middle third.

  Then $\leqslant$ is closed in $C \times C$, but $\mathbb{C} = \langle C; \leqslant, \mathcal{T}\rangle$ is not a Priestley space.
- You can't draw the Stralka space.

A subtlety

- If $\mathbf{X} = \langle X; \leqslant, \mathcal{T}\rangle$ is a Priestley space, then $\leqslant$ is a topologically closed subset of $X \times X$.
- [A. Stralka] Let $\mathbb{C} = \langle C; \leqslant, \mathcal{T}\rangle$, where
  - $\langle C; \mathcal{T}\rangle$ is the Cantor space created by successively deleting middle thirds of the unit interval, and
  - $x < y$ if and only if x and y are the endpoints of a deleted middle third.

  Then $\leqslant$ is closed in $C \times C$, but $\mathbb{C} = \langle C; \leqslant, \mathcal{T}\rangle$ is not a Priestley space.
- You can't draw the Stralka space.
- [Bezhanishvili, Mines, Morandi] Any ordered compact space $\mathbf{X}$ that you can draw in which $\leqslant$ is topologically closed in $X \times X$ is a Priestley space.

A subtlety

- If $\mathbf{X} = \langle X; \leqslant, \mathcal{T}\rangle$ is a Priestley space, then $\leqslant$ is a topologically closed subset of $X \times X$.
- [A. Stralka] Let $\mathbb{C} = \langle C; \leqslant, \mathcal{T}\rangle$, where
  - $\langle C; \mathcal{T}\rangle$ is the Cantor space created by successively deleting middle thirds of the unit interval, and
  - $x < y$ if and only if x and y are the endpoints of a deleted middle third.

  Then $\leqslant$ is closed in $C \times C$, but $\mathbb{C} = \langle C; \leqslant, \mathcal{T}\rangle$ is not a Priestley space.
- You can't draw the Stralka space.
- [Bezhanishvili, Mines, Morandi] Any ordered compact space $\mathbf{X}$ that you can draw in which $\leqslant$ is topologically closed in $X \times X$ is a Priestley space.

Outline

The translation industry: restricted Priestley duals

- Since $\mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ is isomorphic to the lattice $\mathcal{U}^{\mathcal{T}}(\mathbf{X})$ of clopen up-sets of $\mathbf{X}$, it is common to define the dual $E(\mathbf{X})$ of a Priestley space $\mathbf{X}$ to be $\mathcal{U}^{\mathcal{T}}(\mathbf{X})$.

The translation industry: restricted Priestley duals

- Since $\mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ is isomorphic to the lattice $\mathcal{U}^{\mathcal{T}}(\mathbf{X})$ of clopen up-sets of $\mathbf{X}$, it is common to define the dual $E(\mathbf{X})$ of a Priestley space $\mathbf{X}$ to be $\mathcal{U}^{\mathcal{T}}(\mathbf{X})$.

- Hence when translating properties of distributive lattices into properties of Priestley spaces, it is common to use clopen up-sets (or their complements, i.e., clopen down-sets).

The translation industry: restricted Priestley duals

Examples: p-algebras

Let $\mathbf{X}$ and $\mathbf{Y}$ be a Priestley spaces and let $\varphi\colon \mathbf{X} \to \mathbf{Y}$ be continuous and order-preserving.

The translation industry: restricted Priestley duals

Examples: p-algebras

Let $\mathbf{X}$ and $\mathbf{Y}$ be a Priestley spaces and let $\varphi\colon \mathbf{X} \to \mathbf{Y}$ be continuous and order-preserving.

- $\mathbf{X}$ is the dual of a distributive p-algebra, and called a p-space, iff
  - ${\downarrow}U$ is clopen, for every clopen up-set U;

  then $U^* = X \backslash {\downarrow}U$ in $\mathcal{U}^{\mathcal{T}}(\mathbf{X})$.

The translation industry: restricted Priestley duals

Examples: p-algebras

Let **X** and **Y** be a Priestley spaces and let $\varphi\colon \mathbf{X} \to \mathbf{Y}$ be continuous and order-preserving.

- **X** is the dual of a distributive p-algebra, and called a p-space, iff
  - ${\downarrow}U$ is clopen, for every clopen up-set U;

  then $U^* = X \backslash {\downarrow}U$ in $\mathcal{U}^{\mathcal{T}}(\mathbf{X})$.
- If **X** and **Y** are p-spaces, then $\varphi\colon \mathbf{X} \to \mathbf{Y}$ is the dual of a p-algebra homomorphism iff
  - $\varphi(\max(x)) = \max(\varphi(x))$, for all $x \in X$.

  (Here $\max(z)$ denotes the set of maximal elements in ${\uparrow}z$.)

The translation industry: restricted Priestley duals

Examples: Heyting algebras

Let **X** and **Y** be a Priestley spaces and let $\varphi\colon \mathbf{X} \to \mathbf{Y}$ be continuous and order-preserving.

- **X** is the dual of a Heyting algebra, and called a Heyting-space (or Esakia space), iff
  - ${\downarrow}U$ is open, for every open up-set U;

  then $U \to V = X \setminus {\downarrow}(U \backslash V)$ in $\mathcal{U}^{\mathcal{T}}(\mathbf{X})$.

The translation industry: restricted Priestley duals

Examples: Heyting algebras

Let **X** and **Y** be a Priestley spaces and let $\varphi\colon \mathbf{X} \to \mathbf{Y}$ be continuous and order-preserving.

- **X** is the dual of a Heyting algebra, and called a Heyting-space (or Esakia space), iff
  - ${\downarrow}U$ is open, for every open up-set U;

  then $U \to V = X \setminus {\downarrow}(U \backslash V)$ in $\mathcal{U}^{\mathcal{T}}(\mathbf{X})$.
- If **X** and **Y** are Heyting-spaces, then $\varphi\colon \mathbf{X} \to \mathbf{Y}$ is the dual of a Heyting algebra homomorphism iff
  - $\varphi({\uparrow}x) = {\uparrow}\varphi(x)$, for all $x \in X$.

The translation industry: restricted Priestley duals

Examples: Ockham algebras

- $\mathbf{A} = \langle A; \vee, \wedge, g, 0, 1\rangle$ is an Ockham algebra if $\mathbf{A}^\flat := \langle A; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and g satisfies De Morgan's laws and is Boolean complement on $\{0, 1\}$; in symbols,

$$g(a\vee b) = g(a)\wedge g(b),\ \ g(a\wedge b) = g(a)\vee g(b),\ \ g(0) = 1,\ \ g(1) = 0,$$

i.e., g is a lattice-dual endomorphism of $\mathbf{A}^\flat$.

The translation industry: restricted Priestley duals

Examples: Ockham algebras

- $\mathbf{A} = \langle A; \vee, \wedge, g, 0, 1\rangle$ is an Ockham algebra if $\mathbf{A}^\flat := \langle A; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and g satisfies De Morgan's laws and is Boolean complement on $\{0, 1\}$; in symbols,

$$g(a \vee b) = g(a) \wedge g(b), \;\; g(a \wedge b) = g(a) \vee g(b), \;\; g(0) = 1, \;\; g(1) = 0,$$

  i.e., g is a lattice-dual endomorphism of $\mathbf{A}^\flat$.
- Thus $\mathbf{X} = \langle X; \widehat{g}, \leqslant, \mathfrak{T}\rangle$ will be the restricted Priestley dual of an Ockham algebra, known as an Ockham space, if $\mathbf{X}^\flat := \langle X; \leqslant, \mathfrak{T}\rangle$ is a Priestley space and $\widehat{g}\colon X \to X$ is an order-dual endomorphism of $\mathbf{X}^\flat$.

The translation industry: restricted Priestley duals

Examples: Ockham algebras

- $\mathbf{A} = \langle A; \vee, \wedge, g, 0, 1\rangle$ is an Ockham algebra if $\mathbf{A}^\flat := \langle A; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and g satisfies De Morgan's laws and is Boolean complement on $\{0, 1\}$; in symbols,

$$g(a \vee b) = g(a) \wedge g(b), \;\; g(a \wedge b) = g(a) \vee g(b), \;\; g(0) = 1, \;\; g(1) = 0,$$

  i.e., g is a lattice-dual endomorphism of $\mathbf{A}^\flat$.
- Thus $\mathbf{X} = \langle X; \widehat{g}, \leqslant, \mathfrak{T}\rangle$ will be the restricted Priestley dual of an Ockham algebra, known as an Ockham space, if $\mathbf{X}^\flat := \langle X; \leqslant, \mathfrak{T}\rangle$ is a Priestley space and $\widehat{g}\colon X \to X$ is an order-dual endomorphism of $\mathbf{X}^\flat$.
- If $\mathbf{X}$ and $\mathbf{Y}$ are Ockham spaces, then $\varphi\colon \mathbf{X} \to \mathbf{Y}$ is the dual of an Ockham algebra homomorphism if it is continuous, order-preserving and preserves the unary operation $\widehat{g}$.

Outline

Useful facts about Priestley spaces

Prove each of the following claims. The order-theoretic dual of each statement is also true.

Let $\mathbf{X} = \langle X; \leqslant, \mathfrak{T}\rangle$ be a Priestley space.

(1) The set ${\downarrow}Y := \{x \in X \mid (\exists y \in Y)\ x \leqslant y\}$ is closed in $\mathbf{X}$ provided Y is closed in $\mathbf{X}$. In particular, ${\downarrow}y$ is closed in $\mathbf{X}$, for all $y \in X$.

(2) Every up-directed subset of $\mathbf{X}$ has a least upper bound in $\mathbf{X}$.

(3) The set Min($\mathbf{X}$) of minimal elements of $\mathbf{X}$ is non-empty.

(4) Let Y and Z be disjoint closed subsets of $\mathbf{X}$ such that Y is a down-set and Z is an up-set. Then there is a clopen down-set U with $Y \subseteq U$ and $U \cap Z = \varnothing$.

(5) Y is a closed down-set in $\mathbf{X}$ if and only if Y is an intersection of clopen down-sets.