Lecture 1: an invitation to Priestley duality

Brian A. Davey

TACL 2015 School
Campus of Salerno (Fisciano)
15–19 June 2015

Outline

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Concrete examples of bounded distributive lattices

1. The two-element chain

   $\underline{\mathbf{2}} = \langle \{0, 1\}; \vee, \wedge, 0, 1 \rangle.$

2. All subsets of a set S:

   $\langle \wp(S); \cup, \cap, \varnothing, S \rangle.$

3. Finite or cofinite subsets of $\mathbb{N}$:

   $\langle \wp_{\mathsf{FC}}(\mathbb{N}); \cup, \cap, \varnothing, \mathbb{N} \rangle.$

4. Open subsets of a topological space $\mathbf{X}$:

   $\langle \mathcal{O}(\mathbf{X}); \cup, \cap, \varnothing, X \rangle.$

More examples of bounded distributive lattices

5. $\langle \{T, F\}; \text{or}, \text{and}, \text{F}, \text{T}\rangle$.

6. $\langle \mathbb{N} \cup \{0\}; \text{lcm}, \text{gcd}, 1, 0\rangle$.

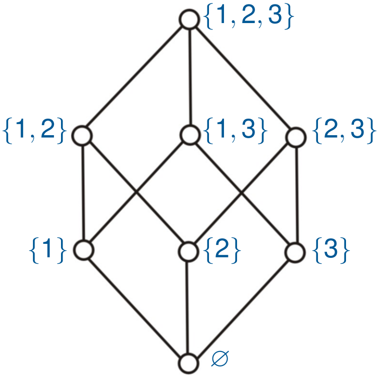
   (Use the fact that, $\text{lcm}(m, n) \cdot \text{gcd}(m, n) = mn$, for all $m, n \in \mathbb{N} \cup \{0\}$, and that a lattice is distributive iff it satisfies $x \vee z = y \vee z \;\; \& \;\; x \wedge z = y \wedge z \implies x = y$.)

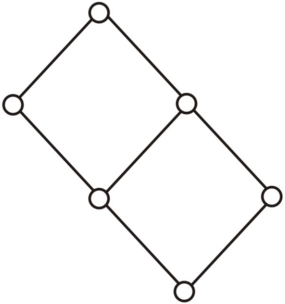
7. Subgroups of a cyclic group **G**,

   $\langle \text{Sub}(\mathbf{G}); \vee, \cap, \{e\}, G\rangle$, where $H \vee K := \text{sg}_{\mathbf{G}}(H \cup K)$.

Drawing distributive lattices

Any distributive lattice $\langle L; \vee, \wedge, 0, 1\rangle$ has a natural order corresponding to set inclusion: $\quad a \leqslant b \iff a \vee b = b$.





$\vee$ union
$\wedge$ intersection
$\leqslant$ inclusion

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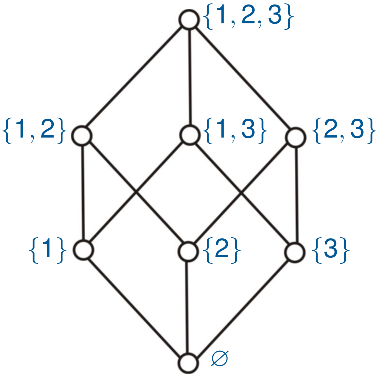

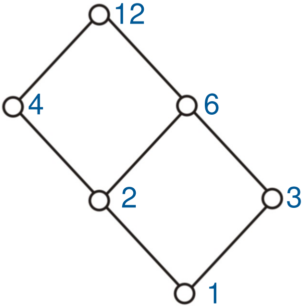

$\vee$ union
$\wedge$ intersection
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$\vee$ lcm
$\wedge$ gcd
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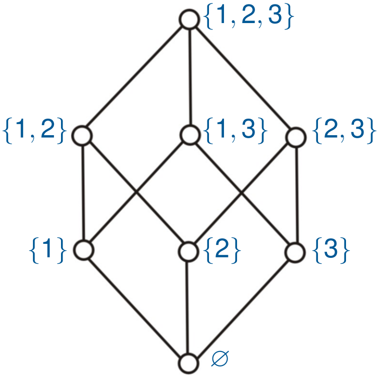
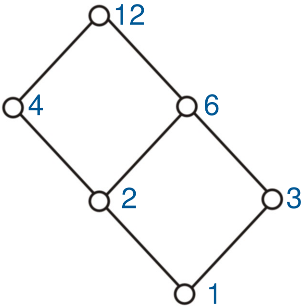


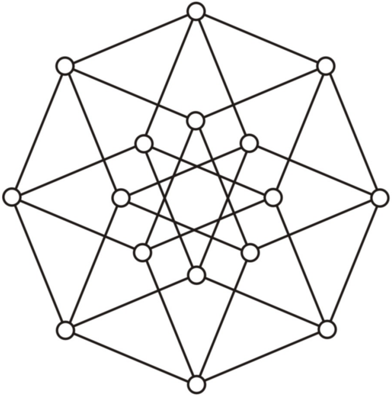


$\vee$ union
$\wedge$ intersection
$\leqslant$ inclusion

$\vee$ lcm
$\wedge$ gcd
$\leqslant$ division

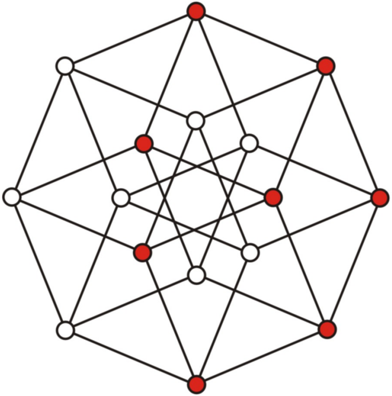
$\vee$ max
$\wedge$ min
$\leqslant$ usual

A sublattice **L** of $\underline{\mathbf{2}}^4$



$\underline{\mathbf{2}}^4$

A sublattice $\mathbf{L}$ of $\underline{\mathbf{2}}^4$



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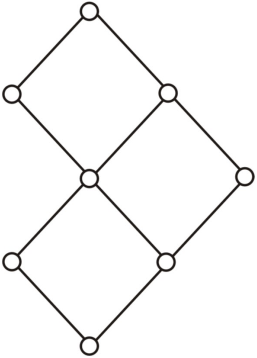

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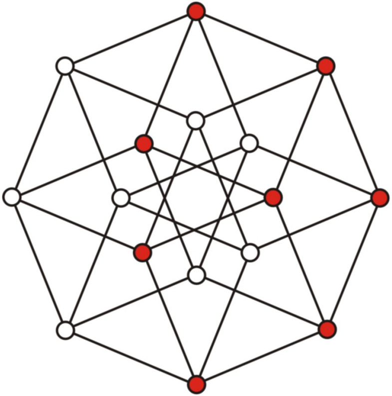

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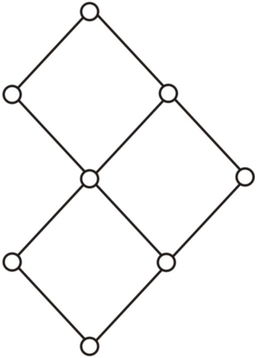

Note: Every distributive lattice embeds into $\underline{\mathbf{2}}^S$, for some set S.

Outline

Representing finite distributive lattices

Birkhoff's representation for a finite distributive lattice **L**

Let **L** be a finite distributive lattice.

L is isomorphic to the collection $\mathcal{O}(\mathbf{P})$ of all down-sets of an ordered set $\mathbf{P} = \langle P; \leqslant \rangle$, under union, intersection, $\varnothing$ and P.

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In fact, we can choose $\mathbf{P}$ to be the ordered set $\langle \mathcal{J}(\mathbf{L}); \leqslant \rangle$ of join-irreducible elements of $\mathbf{L}$.

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Theorem [G. Birkhoff]

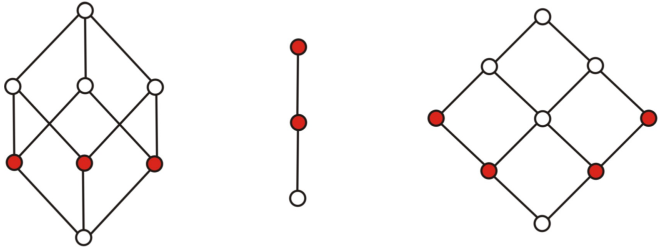
Let $\mathbf{L}$ be a finite distributive lattice and let $\mathbf{P}$ be a finite ordered set. Then

- $\mathbf{L}$ *is isomorphic to* $\mathcal{O}(\mathcal{J}(\mathbf{L}))$*, and*
- $\mathbf{P}$ *is isomorphic to* $\mathcal{J}(\mathcal{O}(\mathbf{P}))$*.*

More examples

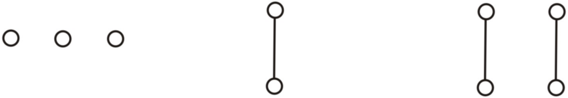
Distributive lattice

$\mathbf{L} \cong \mathcal{O}(\mathcal{J}(\mathbf{L}))$



Ordered set

$\mathbf{P} \cong \mathcal{J}(\mathcal{O}(\mathbf{P}))$



Duality for finite distributive lattices

The classes of

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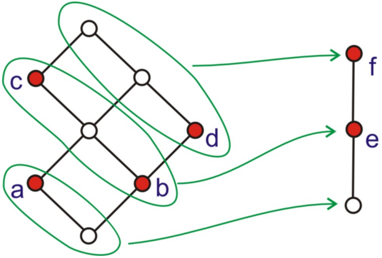

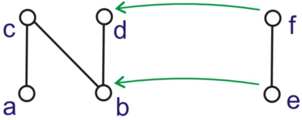

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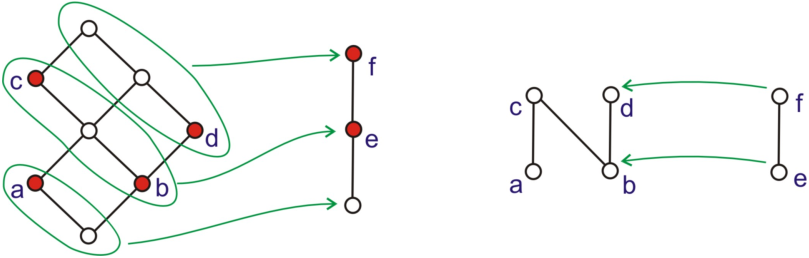

surjections	$\longleftrightarrow$	embeddings
embeddings	$\longleftrightarrow$	surjections
products	$\longleftrightarrow$	disjoint unions

Outline

Duals of finite bounded distributive lattices

Let $\mathbf{L} = \langle L; \vee, \wedge, 0, 1\rangle$ be a finite bounded distributive lattice.

We can define its dual $D(\mathbf{L})$ to be either

- $\mathcal{J}(\mathbf{L})$ — the ordered set of join-irreducible elements of $\mathbf{L}$

or

- $\mathfrak{D}(\mathbf{L}, \underline{\mathbf{2}})$ — the ordered set of $\{0, 1\}$-homomorphisms from $\mathbf{L}$ to the two-element bounded lattice $\underline{\mathbf{2}} = \langle\{0, 1\}; \vee, \wedge, 0, 1\rangle$.

Here $\mathfrak{D}$ denotes the category of bounded distributive lattices.

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In fact, we have the following dual order-isomorphism:

$$\mathcal{J}(\mathbf{L}) \cong^{\partial} \mathfrak{D}(\mathbf{L}, \underline{\mathbf{2}}).$$

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Duals of finite ordered sets

Let $\mathbf{P} = \langle P; \leqslant \rangle$ be a finite ordered set.

We can define its dual $E(\mathbf{P})$ to be either

- $\mathcal{O}(\mathbf{P})$ – the lattice of down-sets (= order ideals) of $\mathbf{P}$

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- $\mathcal{P}(\mathbf{P}, \underset{\sim}{\mathbf{2}})$ – the lattice of order-preserving maps from $\mathbf{P}$ to the two-element ordered set $\underset{\sim}{\mathbf{2}} = \langle \{0, 1\}; \leqslant \rangle$.

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Duals of morphisms

Let **L** and **K** be a finite distributive lattices and let **P** and **Q** be a finite ordered sets.

- There is a bijection between the $\{0,1\}$-homomorphisms from **L** to **K** and the order-preserving maps from $D(\mathbf{K})$ to $D(\mathbf{L})$. Given $f\colon \mathbf{L} \to \mathbf{K}$, we define

$$\varphi\colon \mathcal{J}(\mathbf{K}) \to \mathcal{J}(\mathbf{L}) \text{ by } \varphi(x) := \min(f^{-1}(\uparrow x)),$$
$$\varphi\colon \mathfrak{D}(\mathbf{K}, \underline{\mathbf{2}}) \to \mathfrak{D}(\mathbf{L}, \underline{\mathbf{2}}) \text{ by } \varphi(x) := x \circ f.$$

We denote the map φ by $D(f)$.

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- There is a bijection between the order-preserving maps from **P** to **Q** and the $\{0,1\}$-homomorphisms from $E(\mathbf{Q})$ to $E(\mathbf{P})$. Given $\varphi\colon \mathbf{P} \to \mathbf{Q}$, we define

$$f\colon \mathcal{O}(\mathbf{Q}) \to \mathcal{O}(\mathbf{P}) \text{ by } f(A) := \varphi^{-1}(A),$$
$$f\colon \mathcal{P}(\mathbf{Q}, \underset{\sim}{\mathbf{2}}) \to \mathcal{P}(\mathbf{P}, \underset{\sim}{\mathbf{2}}) \text{ by } f(\alpha) := \alpha \circ \varphi.$$

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The duality at the finite level

- $\underline{\mathbf{2}} = \langle \{0,1\}; \vee, \wedge, 0, 1\rangle$ is the two-element lattice,
- $\underset{\sim}{\mathbf{2}} = \langle \{0,1\}; \leqslant\rangle$ is the two-element ordered set with $0 \leqslant 1$.

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Theorem [G. Birkhoff, H. A. Priestley]

Every finite distributive lattice is encoded by an ordered set:

$$\mathbf{L} \cong ED(\mathbf{L}) \quad \textit{and} \quad \mathbf{P} \cong DE(\mathbf{P}),$$

for each finite distributive lattice $\mathbf{L}$ *and finite ordered set* $\mathbf{P}$.

Indeed, the categories of finite bounded distributive lattices and finite ordered sets are dually equivalent.

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Infinite distributive lattices

Example

The finite-cofinite lattice $\mathbf{L} = \langle \wp_{\mathsf{FC}}(\mathbb{N}); \cup, \cap, \varnothing, \mathbb{N} \rangle$ cannot be obtained as the down-sets of an ordered set.

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- So there would be at least $2^{\mathbb{N}}$ down-sets.

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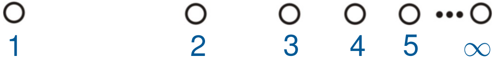
But it can be obtained as the clopen down-sets of a topological ordered set.



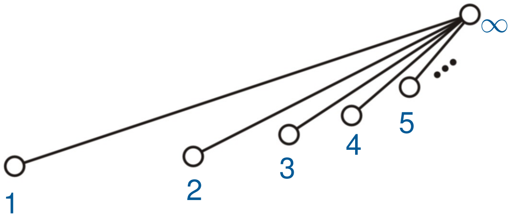
More examples

Distributive lattice:

All finite subsets of $\mathbb{N}$, as well as $\mathbb{N}$ itself,

$$\langle \wp_{\text{fin}}(\mathbb{N}) \cup \{\mathbb{N}\}; \cup, \cap, \varnothing, \mathbb{N}\rangle.$$

Topological ordered set:



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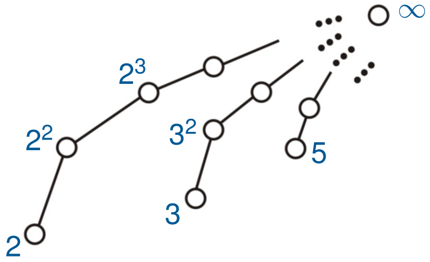

The duality in general

- $\underline{\mathbf{2}} = \langle\{0,1\}; \vee, \wedge, 0, 1\rangle$ is the two-element bounded lattice.

In general, we need to endow the dual $D(\mathbf{L})$ of a bounded distributive lattice $\mathbf{L}$ with a topology. This is easy if we define the dual of $\mathbf{L}$ to be $D(\mathbf{L}) := \mathfrak{D}(\mathbf{L}, \underline{\mathbf{2}})$.

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We first endow $\langle \{0, 1\}; \leqslant \rangle$ with the discrete topology:

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$\mathcal{D}(\mathbf{L}, \underline{\mathbf{2}})$ is a topologically closed subset of $\underset{\sim}{\mathbf{2}}^L$ (easy exercise).

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- $\underline{\mathbf{2}} = \langle \{0,1\}; \vee, \wedge, 0, 1 \rangle$ is the two-element bounded lattice.

In general, we need to endow the dual $D(\mathbf{L})$ of a bounded distributive lattice $\mathbf{L}$ with a topology. This is easy if we define the dual of $\mathbf{L}$ to be $D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}})$.

We first endow $\langle \{0,1\}; \leqslant \rangle$ with the discrete topology:

- $\underset{\sim}{\mathbf{2}} = \langle \{0,1\}; \leqslant, \mathcal{T} \rangle$ is the two-element ordered set with $0 \leqslant 1$ endowed with the discrete topology $\mathcal{T}$.

Then we define

- $D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \leqslant \underset{\sim}{\mathbf{2}}^{L}$.

We put the pointwise order and the product topology on $\underset{\sim}{\mathbf{2}}^{L}$.

Then $D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}})$ inherits its order and topology from $\underset{\sim}{\mathbf{2}}^{L}$.

$\mathcal{D}(\mathbf{L}, \underline{\mathbf{2}})$ is a topologically closed subset of $\underset{\sim}{\mathbf{2}}^{L}$ (easy exercise).

Hence $D(\mathbf{L})$ is a compact ordered topological space.

Priestley spaces

The ordered space $D(\mathbf{L}) := \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \leqslant \underset{\sim}{\mathbf{2}}^{L}$ is more than a compact ordered space. It is a Priestley space.

A topological structure $\mathbf{X} = \langle X; \leqslant, \mathcal{T}\rangle$ is a Priestley space if

- $\langle X; \leqslant, \rangle$ is an ordered set,
- $\mathcal{T}$ is a compact topology on X, and
- for all $x, y \in X$ with $x \nleqslant y$, there is a clopen down-set A of $\mathbf{X}$ such that $x \notin A$ and $y \in A$.

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The category of Priestley spaces is denoted by $\mathcal{P}$.
The following result is very easy to prove.

Lemma

- *$D(\mathbf{L}) = \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \leqslant \underset{\sim}{\mathbf{2}}^{L}$ is a Priestley space, for every bounded distributive lattice.*
- *$E(\mathbf{X}) = \mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{2}}) \leqslant \underline{\mathbf{2}}^{X}$ is a bounded distributive lattice, for every Priestley space $\mathbf{X}$.*

The functors

We now have functors $D\colon \mathcal{D} \to \mathcal{P}$ and $E\colon \mathcal{P} \to \mathcal{D}$ given by

$$D(\mathbf{L}) = \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \leqslant \underset{\sim}{\mathbf{2}}^{L} \quad \text{and} \quad E(\mathbf{X}) = \mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{2}}) \leqslant \underline{\mathbf{2}}^{X}.$$

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D and E are defined on morphisms via composition exactly as they were in the finite case:

- given $f\colon \mathbf{L} \to \mathbf{K}$, we define

$$D(f)\colon \mathcal{D}(\mathbf{K}, \underline{\mathbf{2}}) \to \mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}) \text{ by } D(f)(x) := x \circ f;$$

- given $\varphi\colon \mathbf{X} \to \mathbf{Y}$, we define

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The functors are contravariant as they reverse the direction of the morphisms: $D(f)\colon D(\mathbf{K}) \to D(\mathbf{L})$ and $E(\varphi)\colon E(\mathbf{Y}) \to E(\mathbf{X})$.

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The natural transformations

Let $\mathbf{L} \in \mathcal{D}$ and let $\mathbf{X} \in \mathcal{P}$. There are natural maps

$$e_{\mathbf{L}} \colon \mathbf{L} \to ED(\mathbf{L}) \quad \text{and} \quad \varepsilon_{\mathbf{X}} \colon \mathbf{X} \to DE(\mathbf{X})$$

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- $e_{\mathbf{L}} : \mathbf{L} \to \mathcal{P}(\mathcal{D}(\mathbf{L}, \underline{\mathbf{2}}), \underset{\sim}{\mathbf{2}})$ given by $a \mapsto e_{\mathbf{L}}(a)$,
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Priestley duality tells us that these maps are isomorphisms
(in $\mathcal{D}$ or $\mathcal{P}$, as appropriate).

Priestley duality

Theorem (Priestley duality)

- *The functors* $D\colon \mathfrak{D} \to \mathfrak{P}$ *and* $E\colon \mathfrak{P} \to \mathfrak{D}$ *give a dual category equivalence between* $\mathfrak{D}$ *and* $\mathfrak{P}$.
- *In particular,* $e_{\mathbf{L}}\colon \mathbf{L} \to ED(\mathbf{L})$ *and* $\varepsilon_{\mathbf{X}}\colon \mathbf{X} \to DE(\mathbf{X})$ *are isomorphisms for all* $\mathbf{L} \in \mathfrak{D}$ *and* $\mathbf{X} \in \mathfrak{P}$.

Outline

Some ordered spaces

The following figure comes from Chapter 11 of
Davey and Priestley: Introduction to Lattices and Order.

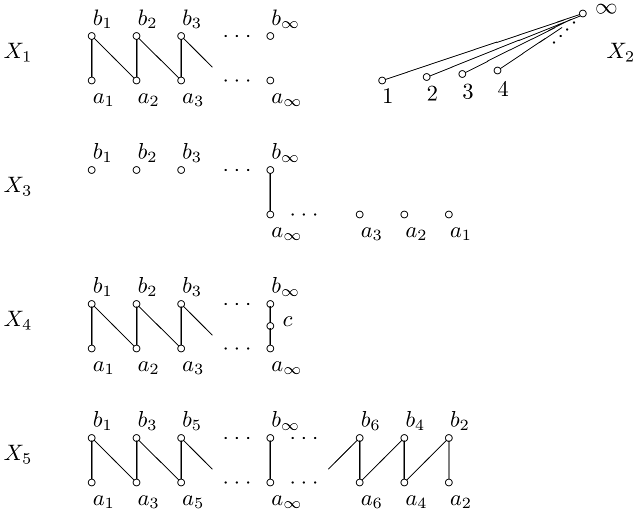

Figure 11.5

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Outline

The translation industry: restricted Priestley duals

- Since $\mathcal{P}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ is isomorphic to the lattice $\mathcal{U}^{\mathcal{T}}(\mathbf{X})$ of clopen up-sets of $\mathbf{X}$, it is common to define the dual $E(\mathbf{X})$ of a Priestley space $\mathbf{X}$ to be $\mathcal{U}^{\mathcal{T}}(\mathbf{X})$.

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- Hence when translating properties of distributive lattices into properties of Priestley spaces, it is common to use clopen up-sets (or their complements, i.e., clopen down-sets).

The translation industry: restricted Priestley duals

Examples: p-algebras

Let **X** and **Y** be a Priestley spaces and let $\varphi\colon \mathbf{X} \to \mathbf{Y}$ be continuous and order-preserving.

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- **X** is the dual of a distributive p-algebra, and called a p-space, iff
  - $\downarrow U$ is clopen, for every clopen up-set U;

  then $U^* = X \setminus \downarrow U$ in $\mathcal{U}^{\mathfrak{T}}(\mathbf{X})$.

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- If **X** and **Y** are p-spaces, then $\varphi\colon \mathbf{X} \to \mathbf{Y}$ is the dual of a p-algebra homomorphism iff
  - $\varphi(\max(x)) = \max(\varphi(x))$, for all $x \in X$.

  (Here $\max(z)$ denotes the set of maximal elements in $\uparrow z$.)

The translation industry: restricted Priestley duals

Examples: Heyting algebras

Let $\mathbf{X}$ and $\mathbf{Y}$ be a Priestley spaces and let $\varphi\colon \mathbf{X} \to \mathbf{Y}$ be continuous and order-preserving.

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The translation industry: restricted Priestley duals

Examples: Ockham algebras

- $\mathbf{A} = \langle A; \vee, \wedge, g, 0, 1\rangle$ is an Ockham algebra if $\mathbf{A}^\flat := \langle A; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and g satisfies De Morgan's laws and is Boolean complement on $\{0, 1\}$; in symbols,

$$g(a \vee b) = g(a) \wedge g(b),\ \ g(a \wedge b) = g(a) \vee g(b),\ \ g(0) = 1,\ \ g(1) = 0,$$

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  i.e., g is a lattice-dual endomorphism of $\mathbf{A}^\flat$.
- Thus $\mathbf{X} = \langle X; \widehat{g}, \leqslant, \mathcal{T}\rangle$ will be the restricted Priestley dual of an Ockham algebra, known as an Ockham space, if $\mathbf{X}^\flat := \langle X; \leqslant, \mathcal{T}\rangle$ is a Priestley space and $\widehat{g} \colon X \to X$ is an order-dual endomorphism of $\mathbf{X}^\flat$.

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- If $\mathbf{X}$ and $\mathbf{Y}$ are Ockham spaces, then $\varphi \colon \mathbf{X} \to \mathbf{Y}$ is the dual of an Ockham algebra homomorphism if it is continuous, order-preserving and preserves the unary operation $\widehat{g}$.

Outline

Useful facts about Priestley spaces

Prove each of the following claims. The order-theoretic dual of each statement is also true.

Let $\mathbf{X} = \langle X; \leqslant, \mathcal{T} \rangle$ be a Priestley space.

(1) The set ${\downarrow}Y := \{x \in X \mid (\exists y \in Y)\ x \leqslant y\}$ is closed in $\mathbf{X}$ provided Y is closed in $\mathbf{X}$. In particular, ${\downarrow}y$ is closed in $\mathbf{X}$, for all $y \in X$.

(2) Every up-directed subset of $\mathbf{X}$ has a least upper bound in $\mathbf{X}$.

(3) The set $\mathrm{Min}(\mathbf{X})$ of minimal elements of $\mathbf{X}$ is non-empty.

(4) Let Y and Z be disjoint closed subsets of $\mathbf{X}$ such that Y is a down-set and Z is an up-set. Then there is a clopen down-set U with $Y \subseteq U$ and $U \cap Z = \varnothing$.

(5) Y is a closed down-set in $\mathbf{X}$ if and only if Y is an intersection of clopen down-sets.