THE LONELY PLANET GUIDE TO THE THEORY OF NATURAL DUALITIES

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In 1991, I gave a series of five lectures on the theory of natural dualities at the Summer School on Algebras and Orders in Montréal, Canada. The theory was then eleven years old, although it had its early genesis back in 1974 in my PhD thesis (see [15]), and the series of lectures was a good chance to take stock. For the next eight years, the notes which resulted from those lectures, **Duality Theory on Ten Dollars a Day** [16], became the standard guide to the theory of natural dualities. In December 1991, David Clark came to La Trobe university for three months and we commenced the work which seven years later culminated in the publication of our text **Natural Dualities for the Working Algebraist** [7]. Below you will find a slightly modified copy of the introduction from *Duality Theory on Ten Dollars a Day* followed by extracts from a number of chapters of *Natural Dualities for the Working Algebraist*[1] reproduced here with the kind permission of Cambridge University Press. I have used the subsection numbers found in the text.

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> En route, I will mention some developments that have occurred in the 17 years since the text was published. Comments on the problems repeated here will be given in footnotes. Other comments will appear in update boxes like this one.

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0. Duality Theory on Ten Dollars a Day

Many young travellers in the realm of general algebra find that the signposts along the road to duality theory point in directions which they would not, of their own accord, choose to travel: to the limits of category theory, to topology's tortuous terrain, to the myriad byways of unfamiliar examples. For them, and perhaps for a few of the not-so-young, we offer this traveller's guide. Here they will find a low cost yet comprehensive tour of the field, avoiding category theory and keeping excursions into topology to a minimum. Our tour is aimed at beginning graduate students who have already completed a first course in topology (up to compactness) and a first course in general algebra (up to Birkhoff's theorems on free algebras, varieties and subdirect representations). Those who would prefer a more comprehensive guide book, including the category-theoretic requisites as well as examples of dualities in action, are referred to the monograph Clark and Davey [7] from which the later sections of these notes are extracted.

To ensure that we are in shape for the longer journey into general algebra, we begin with three day trips into more familiar territory: abelian groups, Boolean algebras and distributive lattices. The reader, and especially the first-time traveller, is warned that the commentary during the guided tour of abelian groups will contain a lot of important chit-chat which will not be repeated during the other two trips.

Abelian Groups. Denote the class of abelian groups by $\mathcal{A}$. The **circle group** is the subgroup $T := \{ z \in \mathbb{C} : |z| = 1 \}$ of the group of nonzero complex numbers under multiplication. For each abelian group $\mathbf{A}$ we denote the set of all homomorphisms $x\colon \mathbf{A} \to \mathbf{T}$ by $\mathcal{A}(\mathbf{A}, \mathbf{T})$. As will soon become apparent, such **homsets** play a crucial role in duality theory. There is a natural map

$$e_{\mathbf{A}}\colon \mathbf{A} \to \mathbf{T}^{\mathcal{A}(\mathbf{A},\mathbf{T})}, \text{ given by } e_{\mathbf{A}}(a)(x) := x(a)$$

for all $a \in A$ and all $x \in \mathcal{A}(\mathbf{A}, \mathbf{T})$. We say that the map $e_{\mathbf{A}}$ is **given by evaluation** since, for each $a \in A$, the map $e_{\mathbf{A}}(a)\colon \mathcal{A}(\mathbf{A}, \mathbf{T}) \to T$ is given by the rule *"evaluate at a"*. It is easily seen that e is a homomorphism. Indeed, since each $x \in \mathcal{A}(\mathbf{A}, \mathbf{T})$ is a homomorphism and since the operation on a power of $\mathbf{T}$ is pointwise, we have

$$e_{\mathbf{A}}(a \cdot b)(x) = x(a \cdot b) = x(a) \cdot x(b) = e_{\mathbf{A}}(a)(x) \cdot e_{\mathbf{A}}(b)(x) = (e_{\mathbf{A}}(a) \cdot e_{\mathbf{A}}(b))(x)$$

for each $x \in \mathcal{A}(\mathbf{A}, \mathbf{T})$ and hence $e_{\mathbf{A}}(a \cdot b) = e_{\mathbf{A}}(a) \cdot e_{\mathbf{A}}(b)$ for all $a, b \in A$. It is a fundamental fact about abelian groups that, if $\mathbf{A} \in \mathcal{A}$ and $a, b \in A$ with $a \neq b$, then there exists a homomorphism $x\colon \mathbf{A} \to \mathbf{T}$ with $x(a) \neq x(b)$. In other words, if $a \neq b$ in A, then there exists $x \in \mathcal{A}(\mathbf{A}, \mathbf{T})$ such that $e_{\mathbf{A}}(a)(x) \neq e_{\mathbf{A}}(b)(x)$ and thus $e_{\mathbf{A}}(a) \neq e_{\mathbf{A}}(b)$. Hence e is an embedding. Consequently, every abelian group is isomorphic to a subgroup of a power of $\mathbf{T}$. Using the usual class operators, $\mathbb{I}$

(all isomorphic copies of), $\mathbb{S}$ (all subgroups of) and $\mathbb{P}$ (all products of), we have $\mathcal{A} = \mathbb{ISP}(\mathbf{T})$.

Thus we have represented each abelian group $\mathbf{A}$ as a group of functions: the group $\mathbf{A}$ is isomorphic to the group

$$\{\, e_{\mathbf{A}}(a) \colon \mathcal{A}(\mathbf{A},\mathbf{T}) \to T \mid a \in A \,\} \leqslant \mathbf{T}^{\mathcal{A}(\mathbf{A},\mathbf{T})}$$

of evaluations maps. This representation would be greatly strengthened if we had some intrinsic description of the evaluation maps. We wish to find some property (expressed in terms of the sets $\mathcal{A}(\mathbf{A},\mathbf{T})$ and T with no reference to the elements of A) which will distinguish the evaluation maps within the set of all maps $\varphi \colon \mathcal{A}(\mathbf{A},\mathbf{T}) \to T$. This search is at the heart of the theory of natural dualities.

First note that T inherits a topology $\mathcal{T}$ from $\mathbb{C}$. In fact, $\langle T; \cdot, ^{-1}, 1, \mathcal{T}\rangle$ is a compact topological group. We impose the product topology on the power T^A: sets of the form

$$U_{a,V} := \{\, u \colon A \to T \mid u(a) \in V \,\},$$

where $a \in A$ and V is open in T, form a subbase for the product topology on T^A. By Tychonoff's Theorem (a product of compact spaces is compact), T^A is compact.

It is an easy exercise to see that the set $\mathcal{A}(\mathbf{A},\mathbf{T})$ of homomorphisms is a closed subspace of T^A. Let $u \colon A \to T$ and assume that u is not a homomorphism. Thus there exist $a, b \in A$ such that $u(a \cdot b) \neq u(a) \cdot u(b)$. Since the topology on T is Hausdorff, there exist open sets V and W in T such that $u(a \cdot b) \in V$, $u(a) \cdot u(b) \in W$ and $V \cap W = \varnothing$. Since multiplication on $\mathbf{T}$ is continuous, there exist open sets W_a and W_b in T such that $u(a) \in W_a$, $u(b) \in W_b$ and $W_a \cdot W_b \subseteq W$. Thus

$$U := U_{ab,V} \cap U_{a,W_a} \cap U_{b,W_b}$$

is an open set in T^A which contains u. Moreover, if $v \in U$, then $v(a \cdot b) \in V$ and $v(a) \cdot v(b) \in W_a \cdot W_b \subseteq W$, whence $v(a \cdot b) \neq v(a) \cdot v(b)$ as $V \cap W = \varnothing$. Thus U is disjoint from $\mathcal{A}(\mathbf{A},\mathbf{T})$ and hence $\mathcal{A}(\mathbf{A},\mathbf{T})$ is closed in T^A.

Since $\mathcal{A}(\mathbf{A},\mathbf{T})$ is a closed subspace of the compact space T^A, it follows immediately that $\mathcal{A}(\mathbf{A},\mathbf{T})$ is also compact. It is a triviality that the evaluation maps are continuous with respect to this topology: if V is open in T and $a \in A$, then

$$\begin{aligned} e(a)^{-1}(V) &= \{\, x \in \mathcal{A}(\mathbf{A},\mathbf{T}) \mid e(a)(x) \in V \,\} \\ &= \{\, x \in \mathcal{A}(\mathbf{A},\mathbf{T}) \mid x(a) \in V \,\} \\ &= \mathcal{A}(\mathbf{A},\mathbf{T}) \cap U_{a,V} \end{aligned}$$

which is open in $\mathcal{A}(\mathbf{A},\mathbf{T})$. Nevertheless, the evaluation maps are not the only continuous maps from $\mathcal{A}(\mathbf{A},\mathbf{T})$ to T.

In order to distinguish the evaluation maps we must impose further structure on T and on $\mathcal{A}(\mathbf{A},\mathbf{T})$. Note that $\mathcal{A}(\mathbf{A},\mathbf{T})$ is closed under the pointwise multiplication: if $x, y \in \mathcal{A}(\mathbf{A},\mathbf{T})$, then $x \cdot y \in \mathcal{A}(\mathbf{A},\mathbf{T})$ since, for all $a, b \in A$,

$$\begin{aligned}(x \cdot y)(a \cdot b) &= x(a \cdot b) \cdot y(a \cdot b)\\ &= (x(a) \cdot x(b)) \cdot (y(a) \cdot y(b))\\ &= (x(a) \cdot y(a)) \cdot (x(b) \cdot y(b))\\ &= (x \cdot y)(a) \cdot (x \cdot y)(b)\end{aligned}$$

which says that $x \cdot y$ is a homomorphism. The crucial identity in this calculation is

$$(s \cdot t) \cdot (u \cdot v) = (s \cdot u) \cdot (t \cdot v)$$

which says precisely that multiplication on $\mathbf{T}$, regarded as a map from $\mathbf{T}^2$ to $\mathbf{T}$, is a homomorphism. The set $\mathcal{A}(\mathbf{A},\mathbf{T})$ contains the identity element of $\mathbf{T}^A$, namely the constant map onto $\{1\}$, because $\{1\}$ is a one-element subgroup of $\mathbf{T}$. Finally, $\mathcal{A}(\mathbf{A},\mathbf{T})$ is closed under forming inverses: if $x\colon \mathbf{A} \to \mathbf{T}$ is a homomorphism, then $x^{-1}\colon \mathbf{A} \to \mathbf{T}$ (defined pointwise) is also a homomorphism since, for all $a, b \in A$,

$$x^{-1}(a \cdot b) = x(a \cdot b)^{-1} = (x(a) \cdot x(b))^{-1} = x(a)^{-1} \cdot x(b)^{-1} = x^{-1}(a) \cdot x^{-1}(b).$$

Again, the crucial identity, namely $(s \cdot t)^{-1} = s^{-1} \cdot t^{-1}$, which holds since $\mathbf{T}$ is abelian, says precisely that $^{-1}\colon \mathbf{T} \to \mathbf{T}$ is a homomorphism. Thus $\mathcal{A}(\mathbf{A},\mathbf{T})$ is a subgroup of $\mathbf{T}^A$, and so we may add this natural pointwise group structure to the topology on $\mathcal{A}(\mathbf{A},\mathbf{T})$. Once more it is trivial that the evaluation maps preserve the additional structure. The evaluation $e(a)\colon \mathcal{A}(\mathbf{A},\mathbf{T}) \to \mathbf{T}$ is a homomorphism for each $a \in A$ since

$$e_{\mathbf{A}}(a)(x \cdot y) = (x \cdot y)(a) = x(a) \cdot y(a) = e_{\mathbf{A}}(a)(x) \cdot e_{\mathbf{A}}(a)(y)$$

for all $x, y \in \mathcal{A}(\mathbf{A},\mathbf{T})$.

To summarise, $\mathcal{A}(\mathbf{A},\mathbf{T})$ is a closed subgroup of $\mathbf{T}^A$ (and hence is a compact topological group) and, for each $a \in A$, the evaluation map $e_{\mathbf{A}}(a)\colon \mathcal{A}(\mathbf{A},\mathbf{T}) \to \mathbf{T}$ is a continuous homomorphism. It is a surprising and highly nontrivial result that the evaluation maps are the only continuous homomorphisms from $\mathcal{A}(\mathbf{A},\mathbf{T})$ to $\mathbf{T}$. This is part of the Pontryagin duality for locally compact abelian groups [67, 68]. Hence, in the case of abelian groups, we have found a natural intrinsic structure on $\mathcal{A}(\mathbf{A},\mathbf{T})$ and $\mathbf{T}$—both are (compact) topological abelian groups—which distinguishes the evaluation maps. Thus every abelian group is isomorphic to the group of all continuous homomorphisms from some compact topological abelian group into the circle group $\mathbf{T}$. The compact topological abelian group $\mathcal{A}(\mathbf{A},\mathbf{T})$ is called the **dual of** $\mathbf{A}$. We shall denote it by $D(\mathbf{A})$.

At this point it is important to draw the distinction between a **representation theory** for a class $\mathcal{A}$ of algebras and a **duality theory** for $\mathcal{A}$. What we have described so far is a representation theory for the class $\mathcal{A}$ of abelian groups. To lift this up to a duality theory for $\mathcal{A}$ we must show that the representation respects homomorphisms (while turning them on their heads). If $\mathbf{A}, \mathbf{B} \in \mathcal{A}$ and $u\colon \mathbf{A} \to \mathbf{B}$ is a homomorphism, then the **dual of** u is the natural map $D(u)\colon D(\mathbf{B}) \to D(\mathbf{A})$ defined by *"compose on the right with u"*, that is,

$$D(u)(x) := x \circ u \in D(\mathbf{A}) = \mathcal{A}(\mathbf{A}, \mathbf{T}) \ \text{ for all } \ x \in D(\mathbf{B}) = \mathcal{A}(\mathbf{B}, \mathbf{T}).$$

The map $D(u)$ is continuous since, if V is open in T and $a \in A$, then

$$D(u)^{-1}(U_{a,V} \cap \mathcal{A}(\mathbf{A}, \mathbf{T})) = U_{u(a),V} \cap \mathcal{A}(\mathbf{B}, \mathbf{T}).$$

Moreover, $D(u)$ is a homomorphism since, for all $x, y \in \mathcal{A}(\mathbf{B}, \mathbf{T})$ and all $a \in A$,

$$\begin{aligned} D(u)(x \cdot y)(a) &= ((x \cdot y) \circ u)(a) = (x \cdot y)(u(a)) = x(u(a)) \cdot y(u(a)) \\ &= (x \circ u)(a) \cdot (y \circ u)(a) = ((x \circ u) \cdot (y \circ u))(a) \\ &= (D(u)(x) \cdot D(u)(y))(a) \end{aligned}$$

whence $\quad D(u)(x \cdot y) = D(u)(x) \cdot D(u)(y).$

The picture we have painted so far during this brief excursion into Pontryagin duality has been intentionally one-sided. We commenced the trip with the cultural mind set of an algebraist for which we make no apology. Nevertheless, since the total picture is highly symmetrical, the other side warrants fuller description.

The duals $D(\mathbf{A})$ for $\mathbf{A} \in \mathcal{A}$ need a home. Since, by construction, each $D(\mathbf{A}) = \mathcal{A}(\mathbf{A}, \mathbf{T})$ is a closed subgroup of a power of $\mathbf{T}$ (regarded as a topological group), a natural choice for their home is the class $\mathbb{IS}_c\mathbb{P}^+(\mathbf{T})$ of all isomorphic copies of closed subgroups of non-trivial powers of $\mathbf{T}$. (A map will be an isomorphism in this context if it is simultaneously a group isomorphism and a topological homeomorphism.) Another natural choice would be the class $\mathcal{X}$ of all compact topological abelian groups. In fact, $\mathcal{X} = \mathbb{IS}_c\mathbb{P}^+(\mathbf{T})$ as we shall see once some further notation is established.

For each $\mathbf{X} \in \mathcal{X}$, the homset $\mathcal{X}(\mathbf{X}, \mathbf{T})$, consisting of the continuous homomorphisms from $\mathbf{X}$ to $\mathbf{T}$, is a subgroup of $\mathbf{T}^X$. The proof is identical to the proof given above that $\mathcal{A}(\mathbf{A}, \mathbf{T})$ is a subgroup of $\mathbf{T}^A$ except that we must now observe that

(a) if $\alpha, \beta : \mathbf{X} \to \mathbf{T}$ are continuous then $\alpha \cdot \beta : \mathbf{X} \to \mathbf{T}$ is continuous (since $\cdot\colon \mathbf{T}^2 \to \mathbf{T}$ is continuous),
(b) if $\alpha\colon \mathbf{X} \to \mathbf{T}$ is continuous then $\alpha^{-1}\colon \mathbf{X} \to \mathbf{T}$ is continuous (since ${}^{-1}\colon \mathbf{T} \to \mathbf{T}$ is continuous), and
(c) the constant map from A onto $\{1\} \subseteq T$ is continuous.

Thus $\mathcal{X}(\mathbf{X}, \mathbf{T}) \in \mathcal{A}$. We refer to $\mathcal{X}(\mathbf{X}, \mathbf{T})$ as the **dual** of $\mathbf{X}$ and denote it by $E(\mathbf{X})$. Just as the map $D\colon \mathcal{A} \to \mathcal{X}$ respects homomorphisms, it is very easily seen that the map $E\colon \mathcal{X} \to \mathcal{A}$ respects continuous homomorphisms (modulo turning them on their heads). If $\mathbf{X}, \mathbf{Y} \in \mathcal{X}$ and $\varphi : \mathbf{X} \to \mathbf{Y}$ is a continuous homomorphism, then the **dual of** φ is the natural map $E(\varphi)\colon E(\mathbf{Y}) \to E(\mathbf{X})$ defined by *"compose on the right with* φ*"*, that is,

$$E(\varphi)(\alpha) := \alpha \circ \varphi \in E(\mathbf{X}) = \mathcal{X}(\mathbf{X}, \mathbf{T}) \text{ for all } \alpha \in E(\mathbf{Y}) = \mathcal{X}(\mathbf{Y}, \mathbf{T}).$$

We now have two natural maps given by evaluation: for all $\mathbf{A} \in \mathcal{A}$ and $\mathbf{X} \in \mathcal{X}$,

$$e_{\mathbf{A}}\colon \mathbf{A} \to ED(\mathbf{A}) = \mathcal{X}(D(\mathbf{A}), \mathbf{T}) = \mathcal{X}(\mathcal{A}(\mathbf{A}, \mathbf{T}), \mathbf{T})$$

defined by $e_{\mathbf{A}}(a)(x) := x(a)$ for all $a \in A$ and $x \in \mathcal{A}(\mathbf{A}, \mathbf{T})$, and

$$\varepsilon_{\mathbf{X}}\colon \mathbf{X} \to DE(\mathbf{X}) = \mathcal{A}(E(\mathbf{X}), \mathbf{T}) = \mathcal{A}(\mathcal{X}(\mathbf{X}, \mathbf{T}), \mathbf{T})$$

defined by $\varepsilon_{\mathbf{X}}(x)(\alpha) := \alpha(x)$ for all $x \in X$ and $\alpha \in \mathcal{X}(\mathbf{X}, \mathbf{T})$. While it is clear that $\mathbb{IS}_c\mathbb{P}^+(\mathbf{T}) \subseteq \mathcal{X}$, the reverse inclusion is far from clear. The vital (and difficult) fact is that if $\mathbf{X}$ is a compact topological abelian group then there are enough continuous homomorphisms from $\mathbf{X}$ into $\mathbf{T}$ to separate the points of X, that is, if $x \neq y$ in X, then there exists a continuous homomorphism $\alpha\colon \mathbf{X} \to \mathbf{T}$ such that $\alpha(x) \neq \alpha(y)$. From this it is easily seen that the map $\varepsilon_{\mathbf{X}}\colon \mathbf{X} \to DE(\mathbf{X})$ is an isomorphism of $\mathbf{X}$ onto a closed subgroup of a power of $\mathbf{T}$. Thus $\mathbb{IS}_c\mathbb{P}^+(\mathbf{T})$ is the class of all compact topological abelian groups.

As was discussed earlier, the map $e_{\mathbf{A}}\colon \mathbf{A} \to ED(\mathbf{A})$ is an isomorphism for all $\mathbf{A} \in \mathcal{A}$. This is what we mean when we say that **we have a duality between $\mathcal{A}$ and $\mathcal{X}$**. In many applications this is all that is needed: each $\mathbf{A} \in \mathcal{A}$ has a representation as $E(\mathbf{X})$ for some $\mathbf{X} \in \mathcal{X}$, but $\mathbf{X}$ need not be unique up to isomorphism. If, in addition, the map $\varepsilon_{\mathbf{X}}\colon \mathbf{X} \to DE(\mathbf{X})$ is an isomorphism for all $\mathbf{X} \in \mathcal{X}$, then we say that **the duality between $\mathcal{A}$ and $\mathcal{X}$ is full**. The Pontryagin duality between the class $\mathcal{A}$ of abelian groups and the class $\mathcal{X}$ of compact topological abelian groups is a full duality and hence every abelian group $\mathbf{A}$ can be represented as the group of continuous homomorphisms from a unique-up-to-isomorphism compact topological abelian group into the circle group.

The circle group has a split personality. It lives in $\mathcal{A}$ as the abelian group $\underline{\mathbf{T}} = \langle T; \cdot, ^{-1}, 1\rangle$ and in $\mathcal{X}$ as the compact topological group $\underset{\sim}{\mathbf{T}} = \langle T; \cdot, ^{-1}, 1, \mathcal{T}\rangle$. As we shall see, this schizophrenic behaviour is completely typical within duality theory. In general, our choice of notation will make it clear which class an object belongs to: $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ for groups in $\mathcal{A}$ and $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ for topological groups in $\mathcal{X}$. But, to make it clear which role the circle group is playing, we shall henceforth use the $\underline{\mathbf{T}}$ versus $\underset{\sim}{\mathbf{T}}$ notation.

Boolean Algebras. The dual of a Boolean algebra $\mathbf{A} = \langle A; \vee, \wedge, ', 0, 1\rangle$ is usually defined to be the set $\mathscr{U}(\mathbf{A})$ of ultrafilters of $\mathbf{A}$ endowed with the topology generated by the sets of the form

$$\mathscr{U}_a := \{\, F \in \mathscr{U}(\mathbf{A}) \mid a \in F \,\}$$

for $a \in A$. Note that $\mathscr{U}_a$ is clopen (that is, both closed and open) in this topology since

$$X \backslash \mathscr{U}_a = \{\, F \in \mathscr{U}(\mathbf{A}) \mid a \notin F \,\} = \{\, F \in \mathscr{U}(\mathbf{A}) \mid a' \in F \,\} = \mathscr{U}_{a'}$$

which is a basic open set. Stone's duality for Boolean algebras [74] (or see [36]) asserts, in part, that the map $e : a \mapsto \mathscr{U}_a$ is an isomorphism of $\mathbf{A}$ onto the Boolean algebra of clopen subsets of $\mathscr{U}(\mathbf{A})$. Our task now is to see that this can be expressed naturally in terms of homsets in a manner strictly analogous to what we observed during our day trip into the Pontryagin duality for abelian groups. The role of the circle group $\underline{\mathbf{T}}$ will now be played by the two-element Boolean algebra $\underline{\mathbf{2}} = \langle \{0,1\}; \vee, \wedge, ', 0, 1 \rangle$ while the topological group $\underset{\sim}{\mathrm{T}}$ will be replaced by a much simpler object, namely $\underset{\sim}{2} = \langle \{0,1\}; \mathcal{T} \rangle$, where $\mathcal{T}$ is the discrete topology.

Let $\mathcal{B}$ denote the class of all Boolean algebras. For any subset F of A we define a map $\chi_F : A \to \{0,1\}$, the **characteristic function** of F, by

$$\chi_F(a) := \begin{cases} 1 & \text{if } a \in F, \\ 0 & \text{if } a \notin F. \end{cases}$$

It is easily seen that F is a prime filter of the Boolean algebra $\mathbf{A}$ if and only if χ_F is a lattice homomorphism onto $\underline{\mathbf{2}}$. But a filter of a Boolean algebra $\mathbf{A}$ is prime if and only if it is an ultrafilter, and a lattice homomorphism from $\mathbf{A}$ onto $\underline{\mathbf{2}}$ is automatically a Boolean algebra homomorphism. Thus $\varphi : F \mapsto \chi_F$ is a bijection between the set $\mathscr{U}(\mathbf{A})$ of ultrafilters of $\mathbf{A}$ and the set $\mathcal{B}(\mathbf{A}, \underline{\mathbf{2}})$ of all Boolean algebra homomorphisms $x \colon \mathbf{A} \to \underline{\mathbf{2}}$. A simple modification of the proof for the circle group shows that the natural map

$$e_{\mathbf{A}} \colon \mathbf{A} \to \underline{\mathbf{2}}^{\mathcal{B}(\mathbf{A},\underline{\mathbf{2}})}, \text{ given by } \; e_{\mathbf{A}}(a)(x) := x(a)$$

for all $a \in A$ and all $x \in \mathcal{B}(\mathbf{A}, \underline{\mathbf{2}})$, is a homomorphism. The Boolean Ultrafilter Theorem says precisely that if $a \neq b$ in A, then there exists an ultrafilter F of $\mathbf{A}$ which contains exactly one of a and b. Thus, taking $x = \chi_F$, we have

$$e_{\mathbf{A}}(a)(x) = x(a) = \chi_F(a) \neq \chi_F(b) = x(b) = e_{\mathbf{A}}(b)(x)$$

and consequently $e_{\mathbf{A}}$ is an embedding. Hence $\mathcal{B} = \mathbb{ISP}(\underline{\mathbf{2}})$.

By mimicing the proof for the circle group, it is easily seen that $\mathcal{B}(\mathbf{A}, \underline{\mathbf{2}})$ is a closed subspace of the product space $\underset{\sim}{2}^{A}$. (All that is needed is that the topology on $\underset{\sim}{2}$ is Hausdorff and that the Boolean algebra operations on $\underline{\mathbf{2}}$ are continuous with respect to the topology on $\underset{\sim}{2}$: both are trivially true since the topology on $\underset{\sim}{2}$ is discrete.) Recall that if $a \in A$ and $V \subseteq 2 = \{0,1\}$, then

$$U_{a,V} := \{\, u \colon A \to T \mid u(a) \in V \,\}.$$

Since

$$\varphi(\mathscr{U}_a) = U_{a,\{1\}} \cap \mathcal{B}(\mathbf{A}, \underline{\mathbf{2}}) \text{ and } \varphi(\mathscr{U}_{a'}) = U_{a,\{0\}} \cap \mathcal{B}(\mathbf{A}, \underline{\mathbf{2}}),$$

for all $a \in A$, the map $\varphi\colon \mathcal{U}(\mathbf{A}) \to \mathcal{B}(\mathbf{A}, \underline{\mathbf{2}})$ is a homeomorphism. Thus we may define the **dual**, $D(\mathbf{A})$, to be the compact topological space $\mathcal{B}(\mathbf{A}, \underline{\mathbf{2}})$ with its topology inherited as a subspace of the power $\underset{\sim}{\mathbf{2}}^A$.

As a home for the dual spaces $D(\mathbf{A})$ for $\mathbf{A} \in \mathcal{B}$ we take the class $\mathcal{Z} := \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\underset{\sim}{\mathbf{2}})$ of all isomorphic (that is, homeomorphic) copies of closed subspaces of non-trivial powers of the two-element discrete space $\underset{\sim}{\mathbf{2}}$. For each $\mathbf{X} \in \mathcal{Z}$, the homset $\mathcal{Z}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$ of all continuous maps from $\mathbf{X}$ into $\underset{\sim}{\mathbf{2}}$ is a subalgebra of $\underline{\mathbf{2}}^X$ and hence $\mathcal{Z}(\mathbf{X}, \underset{\sim}{\mathbf{2}}) \in \mathcal{B}$. Thus we define the **dual** of $\mathbf{X}$ to be $E(\mathbf{X}) := \mathcal{Z}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$, a subalgebra of $\underline{\mathbf{2}}^X$. Note that a subset U of X is clopen if and only if its characteristic function χ_U is continuous and hence $E(\mathbf{X})$ is isomorphic to the Boolean algebra of clopen subsets of $\mathbf{X}$.

We leave it to the reader to define the dual $D(u)\colon D(\mathbf{B}) \to D(\mathbf{A})$ of a homomorphism $u\colon \mathbf{A} \to \mathbf{B}$ and the dual $E(\varphi)\colon E(\mathbf{Y}) \to E(\mathbf{X})$ of a continuous map $\varphi\colon \mathbf{X} \to \mathbf{Y}$. (Just replace $\mathcal{A}$ by $\mathcal{B}$ and $\mathbf{T}$ by either $\underline{\mathbf{2}}$ or $\underset{\sim}{\mathbf{2}}$ in the definition given in the abelian group case.)

As in the abelian group case, we have two natural maps given by evaluation: for all $\mathbf{A} \in \mathcal{B}$ and all $\mathbf{X} \in \mathcal{Z}$,

$$e_{\mathbf{A}}\colon \mathbf{A} \to ED(\mathbf{A}) = \mathcal{Z}(D(\mathbf{A}), \underset{\sim}{\mathbf{2}}) = \mathcal{Z}(\mathcal{B}(\mathbf{A}, \underline{\mathbf{2}}), \underset{\sim}{\mathbf{2}}),$$

defined by $e_{\mathbf{A}}(a)(x) := x(a)$ for all $a \in A$ and $x \in \mathcal{B}(\mathbf{A}, \underline{\mathbf{2}})$, and

$$\varepsilon_{\mathbf{X}}\colon \mathbf{X} \to DE(\mathbf{X}) = \mathcal{B}(E(\mathbf{X}), \underline{\mathbf{2}}) = \mathcal{B}(\mathcal{Z}(\mathbf{X}, \underset{\sim}{\mathbf{2}}), \underline{\mathbf{2}}),$$

defined by $\varepsilon_{\mathbf{X}}(x)(\alpha) := \alpha(x)$ for all $x \in X$ and $\alpha \in \mathcal{Z}(\mathbf{X}, \underset{\sim}{\mathbf{2}})$. The fact that $\mathbf{A}$ is isomorphic to the Boolean algebra of clopen subsets of $\mathcal{U}(\mathbf{A})$ translates into the statement that the map $e_{\mathbf{A}}\colon \mathbf{A} \to ED(\mathbf{A})$ is an isomorphism. In this case, all that is needed to distinguish the evaluation maps is the topology on $\mathcal{B}(\mathbf{A}, \underline{\mathbf{2}})$ and on $\underset{\sim}{\mathbf{2}}$: a map $u\colon \mathcal{B}(\mathbf{A}, \underline{\mathbf{2}}) \to \underset{\sim}{\mathbf{2}}$ is an evaluation map $e(a)$ for some $a \in A$ if and only if it is continuous. Thus we have a **duality** between $\mathcal{B}$ and $\mathcal{Z}$. In fact the duality is **full**, that is, $\varepsilon_{\mathbf{X}}\colon \mathbf{X} \to DE(\mathbf{X})$ is also an isomorphism for each $\mathbf{X} \in \mathcal{Z}$.

It is natural to ask for an axiomatisation of the class $\mathcal{X}$. While not all applications of a duality require an axiomatisation of the dual structures, the utility of the duality is greatly increased if we have such an axiomatisation. It is a very easy exercise to see that $\mathbf{X} \in \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\underset{\sim}{\mathbf{2}})$ if and only if $\mathbf{X}$ is a compact Hausdorff space which is **totally disconnected** (that is, has a basis of clopen sets). Such spaces are referred to as **Stone spaces** or **Boolean spaces**.

That completes our second day trip. Much of what we have seen so far has an air of general algebra about it. We can already take a step back and survey the scene at a higher level. We need a class of algebras of the form $\mathcal{A} = \mathbb{I}\mathbb{S}\mathbb{P}(\underline{\mathbf{M}})$ for some algebra $\underline{\mathbf{M}} = \langle M; F\rangle$. The algebra $\underline{\mathbf{M}}$ should have a compact topology $\mathcal{T}$ with respect to which each operation $f \in F$ is continuous. We will define $\mathcal{X}$ to be $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ and define the dual of $\mathbf{A} \in \mathcal{A}$ to be $D(\mathbf{A}) := \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ regarded as a substructure of $\underset{\sim}{\mathbf{M}}^A$ where $\underset{\sim}{\mathbf{M}} = \langle M; ???, \mathcal{T}\rangle$. Unfortunately, with only two examples under our belts, it is not yet clear what structure will be appropriate on $\underset{\sim}{\mathbf{M}}$. We need an example where the general framework is the same but where the alter ego, $\underset{\sim}{\mathbf{M}}$, of the algebra $\underline{\mathbf{M}}$ has a character quite different from the strongly algebraic nature of $\underset{\sim}{\mathbf{T}} = \langle T; \cdot, ^{-1}, 1, \mathcal{T}\rangle$ and the purely topological nature of $\underset{\sim}{\mathbf{2}} = \langle \{0,1\}; \mathcal{T}\rangle$. Hence we commence our third and final day trip.

Distributive Lattices. Just as the dual of a Boolean algebra is usually defined in terms of ultrafilters, the dual of a distributive lattice $\mathbf{A} = \langle A; \vee, \wedge\rangle$ is usually defined in terms of prime filters. We may define the dual of $\mathbf{A}$ to be the set $\mathscr{F}(\mathbf{A})$ of prime filters of $\mathbf{A}$, where we allow both $\varnothing$ and A as prime filters. As in the Boolean case, we endow $\mathscr{F}(\mathbf{A})$ with a topology $\mathcal{T}$: take the sets

$$\mathscr{F}_a := \{\, F \in \mathscr{F}(\mathbf{A}) \mid a \in F \,\},$$

where $a \in A$, and their complements as a subbase for $\mathcal{T}$. We also order $\mathscr{F}(\mathbf{A})$ by set inclusion. Thus the dual of $\mathbf{A}$ is the bounded, ordered topological space $\langle \mathscr{F}(\mathbf{A}); \varnothing, A, \subseteq, \mathcal{T}\rangle$. According to Priestley's duality for the class $\mathcal{D}$ of distributive lattices [69, 70] (or see [36]), $e_{\mathbf{A}} : a \mapsto \mathscr{F}_a$ is an isomorphism of $\mathbf{A}$ onto the lattice of clopen increasing subsets of $\mathscr{F}(\mathbf{A})$. As in the Boolean case, this translates easily into a statement about homsets and evaluation maps.

We are now in fairly familiar territory. Let $\underline{\mathbf{D}} = \langle \{0,1\}; \vee, \wedge\rangle$ be the two-element distributive lattice. Once again, a very simple calculation shows that the natural map

$$e_{\mathbf{A}} \colon \mathbf{A} \to \underline{\mathbf{D}}^{\mathcal{D}(\mathbf{A},\underline{\mathbf{D}})}, \text{ given by } \; e_{\mathbf{A}}(a)(x) := x(a)$$

for all $a \in A$ and all $x \in \mathcal{D}(\mathbf{A}, \underline{\mathbf{D}})$, is a homomorphism. The Distributive Prime Ideal Theorem guarantees that if $a \neq b$ in A, then there exists a prime filter F which contains exactly one of a and b. Thus the characteristic function $\chi_F : A \to \{0,1\}$ separates a and b. The argument given for Boolean algebras applies without change, whence $e_{\mathbf{A}}$ is an embedding. Consequently, $\mathcal{D} = \mathbb{ISP}(\underline{\mathbf{D}})$.

Let $\underset{\sim}{\mathbf{D}} = \langle \{0,1\}; 0, 1, \leqslant, \mathcal{T}\rangle$ be the two-element chain with the bounds as nullary operations and endowed with the discrete topology. Since $\mathcal{D}(\mathbf{A}, \underline{\mathbf{D}})$ is a closed subspace of $\underset{\sim}{\mathbf{2}}^A$, it inherits both a compact topology, an order and its bounds from the power $\underset{\sim}{\mathbf{2}}^A$. It is a very easy exercise to show that $\varphi : F \mapsto \chi_F$ is a homeomorphism and an order-isomorphism between $\mathscr{F}(\mathbf{A})$ and $\mathcal{D}(\mathbf{A}, \underline{\mathbf{D}})$. Thus we define the **dual** of $\mathbf{A}$ to be the bounded, ordered Boolean space $\mathcal{D}(\mathbf{A}, \underline{\mathbf{D}})$. The algebraic half of Priestley duality can now be reformulated in terms of homsets as: *for every distributive lattice* $\mathbf{A}$*, the evaluation maps* $e_{\mathbf{A}}(a)$ *for* $a \in A$ *are the only continuous,* $\{0,1\}$*-order-preserving maps from* $\mathcal{D}(\mathbf{A}, \underline{\mathbf{D}})$ *into* $\underset{\sim}{\mathbf{2}}$*.*

The natural home for the duals $D(\mathbf{A})$ for $\mathbf{A} \in \mathcal{D}$ is the class $\mathcal{P}_{01} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{D}})$ of all isomorphic (that is, simultaneously homeomorphic and order-isomorphic) copies of closed substructures of non-trivial powers of $\underset{\sim}{\mathbf{D}} = \langle \{0,1\}; 0, 1, \leqslant, \mathcal{T}\rangle$.

For each $\mathbf{X} \in \mathcal{P}_{01}$, the homset $\mathcal{P}_{01}(\mathbf{X}, \underset{\sim}{\mathbf{D}})$ of all continuous, $\{0,1\}$-order-preserving maps from $\mathbf{X}$ into $\underset{\sim}{\mathbf{D}}$ is a sublattice of $\underline{\mathbf{D}}^X$ and thus $\mathcal{P}_{01}(\mathbf{X}, \underset{\sim}{\mathbf{D}}) \in \mathcal{D}$. Although the proof is easy, it is essential for our further travels that we gauge the general-algebraic import of this observation.

Let $\alpha, \beta \in \mathcal{P}_{01}(\mathbf{X}, \underset{\sim}{\mathbf{D}})$. Then $\alpha \vee \beta \colon \mathbf{X} \to \underset{\sim}{\mathbf{D}}$ and $\alpha \wedge \beta \colon \mathbf{X} \to \underset{\sim}{\mathbf{D}}$ are continuous since $\vee \colon 2^2 \to 2$ and $\wedge \colon 2^2 \to 2$ are continuous. The maps $\alpha \vee \beta$ and $\alpha \wedge \beta$ preserve 0 and 1 since $\{0\}$ and $\{1\}$ are sublattices of $\underline{\mathbf{D}}$. Finally, $\alpha \vee \beta$ and $\alpha \wedge \beta$ are order-preserving since, for all $x, y \in X$,

$$\begin{aligned} x \leqslant y &\implies \alpha(x) \leqslant \alpha(y) \ \& \ \beta(x) \leqslant \beta(y) \ \text{ as } \ \alpha, \beta \in \mathcal{P}_{01}(\mathbf{X}, \underset{\sim}{\mathbf{D}}) \\ &\implies \alpha(x) \vee \beta(x) \leqslant \alpha(y) \vee \beta(y) \ \& \ \alpha(x) \wedge \beta(x) \leqslant \alpha(y) \wedge \beta(y) \\ &\implies (\alpha \vee \beta)(x) \leqslant (\alpha \vee \beta)(y) \ \& \ (\alpha \wedge \beta)(x) \leqslant (\alpha \wedge \beta)(y). \end{aligned}$$

This calculation depends upon the fact that $\underline{\mathbf{D}}$ satisfies

$$u \leqslant v \ \& \ s \leqslant t \implies u \vee s \leqslant v \vee t \ \& \ u \wedge s \leqslant v \wedge t$$

or, equivalently, (recalling that $\leqslant$ is a subset of 2^2),

$$(u, v) \in \leqslant \ \& \ (s, t) \in \leqslant \implies (u, v) \vee (s, t) \in \leqslant \ \& \ (u, v) \wedge (s, t) \in \leqslant.$$

This says precisely that $\leqslant$ is a sublattice of $\underline{\mathbf{D}}^2$. Hence, we have used the fact that $\{0\}$ and $\{1\}$ are sublattices of $\underline{\mathbf{D}}$ and that $\leqslant$ is a sublattice of $\underline{\mathbf{D}}^2$.

We define the dual of $\mathbf{X}$ to be $E(\mathbf{X}) := \mathcal{P}_{01}(\mathbf{X}, \underset{\sim}{\mathbf{D}})$, a sublattice of $\underline{\mathbf{D}}^X$. A subset U of X is clopen and increasing if and only if its characteristic function $\chi_U \colon X \to \underset{\sim}{\mathbf{D}}$ is both continuous and order-preserving. Thus $E(\mathbf{X})$ is isomorphic to the lattice of all clopen increasing non-empty, proper, subsets of $\mathbf{X}$.

The maps $D \colon \mathcal{D} \to \mathcal{P}_{01}$ and $E \colon \mathcal{P}_{01} \to \mathcal{D}$ can be defined on morphisms via composition exactly as in the Boolean case. Of course, we once again have the two natural maps given by evaluation: for all $\mathbf{A} \in \mathcal{D}$ and all $\mathbf{X} \in \mathcal{P}_{01}$,

$$e_{\mathbf{A}} \colon \mathbf{A} \to ED(\mathbf{A}) = \mathcal{P}_{01}(D(\mathbf{A}), \underset{\sim}{\mathbf{D}}) = \mathcal{P}_{01}(\mathcal{D}(\mathbf{A}, \underline{\mathbf{D}}), \underset{\sim}{\mathbf{D}}),$$

defined by $e_{\mathbf{A}}(a)(x) := x(a)$ for all $a \in A$ and $x \in \mathcal{D}(\mathbf{A}, \underline{\mathbf{D}})$, and

$$\varepsilon_{\mathbf{X}} \colon \mathbf{X} \to DE(\mathbf{X}) = \mathcal{D}(E(\mathbf{X}), \underline{\mathbf{D}}) = \mathcal{D}(\mathcal{P}_{01}(\mathbf{X}, \underset{\sim}{\mathbf{D}}), \underline{\mathbf{D}}),$$

defined by $\varepsilon_{\mathbf{X}}(x)(\alpha) := \alpha(x)$ for all $x \in X$ and $\alpha \in \mathcal{P}_{01}(\mathbf{X}, \underset{\sim}{\mathbf{D}})$. Priestley duality tells us that we have a **full duality** between $\mathcal{D}$ and $\mathcal{P}_{01}$, that is, $e_{\mathbf{A}} \colon \mathbf{A} \to ED(\mathbf{A})$ and $\varepsilon_{\mathbf{X}} \colon \mathbf{X} \to DE(\mathbf{X})$ are isomorphisms for all $\mathbf{A} \in \mathcal{D}$ and $\mathbf{X} \in \mathcal{X}$.

An ordered topological space $\mathbf{X}$ is called **totally order-disconnected** if, for all $x, y \in X$ with $x \nleqslant y$, there exists a clopen increasing subset U of $\mathbf{X}$ such that $x \in U$ but $y \notin U$. This is precisely the notion needed to axiomatise $\mathcal{P}_{01}$: a bounded, ordered topological space $\mathbf{X}$ is in $\mathcal{P}_{01} = \mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}(\underset{\sim}{\mathbf{D}})$ if and only if $\mathbf{X}$ is compact and totally order-disconnected. Such ordered topological spaces are often called **bounded TODC spaces** or **bounded Priestley spaces**.

Applications of Priestley's duality for $\mathcal{D}$ abound—see, for example, the survey articles Davey and Duffus [18] and Priestley [71].

Having completed our three day-trips, we are now ready to commence our guided tour of general duality theory. But before we do, we should address a fundamental question: *"Why bother?"* There are many reasons for developing a duality (of the type described in this guide) for your favourite class $\mathcal{A}$ of algebras. Here are a few: see [7] for more details.

- Once we have a duality for $\mathcal{A}$ we have a uniform way of representing each algebra $\mathbf{A} \in \mathcal{A}$ as an algebra of continuous functions.
- If we have a full duality and have axiomatised the class $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$, we can find examples of particular algebras in $\mathcal{A}$ by constructing objects in $\mathcal{X}$, which often turns out to be easier.
- Algebraic questions in $\mathcal{A}$ can be answered by translating them into (often simpler) questions in $\mathcal{X}$. For example,
  (1) free algebras in $\mathcal{A}$ are easily described via their duals in $\mathcal{X}$,
  (2) while a coproduct $\mathbf{A} * \mathbf{B}$ is often difficult to describe in $\mathcal{A}$, the dual, $D(\mathbf{A} * \mathbf{B})$, is simply the cartesian product $D(\mathbf{A}) \times D(\mathbf{B})$,
  (3) congruence lattices in $\mathcal{A}$ may be studied by looking at lattices of closed substructures in $\mathcal{X}$,
  (4) injective algebras in $\mathcal{A}$ may be characterised by first studying projective structures in $\mathcal{X}$,
  (5) algebraically closed and existentially closed algebras may be described via their duals.
- Some dualities have the particularly powerful property of being "logarithmic" in that they turn products into sums. For example, for both Boolean algebras and bounded distributive lattices we have $D(\mathbf{A} \times \mathbf{B}) \cong D(\mathbf{A}) \dot\cup D(\mathbf{B})$.

Chapter 1: Dual Equivalences and Where to Find Them

Structured topological spaces A natural duality is a special kind of dual representation that can exist between a finitely generated quasi-variety $\mathcal{A}$ and a category $\mathcal{X}$ of structured Boolean spaces. In this section we describe the kinds of structured Boolean spaces that arise in this way. The quasi-variety of distributive lattices, for example, is dual to a category of Boolean spaces which carry two nullary total operations (constants) and one binary relation. Other dualities will require spaces with different kinds of structure. To lay the groundwork for these dualities we consider the most general kind of structure that we might want to impose. We begin with three sets of symbols:

(i) a set G of finitary total operation symbols,
(ii) a set H of finitary partial operation symbols,
(iii) a set R of finitary relation symbols.

Each symbol carries an arity, which is a natural number defined by a fixed arity function on these sets of symbols. Operations may be nullary, but partial operations and relations must have positive arities. By a **structured topological space** of type $\langle G, H, R\rangle$ we mean a structure

$$\mathbf{X} = \langle X; G^{\mathbf{X}}, H^{\mathbf{X}}, R^{\mathbf{X}}, \mathcal{T}^{\mathbf{X}}\rangle$$

where

(i) $G^{\mathbf{X}}$ consists of an n-ary total operation $g^{\mathbf{X}} \colon X^n \to X$ for each n-ary total operation symbol $g \in G$,

(ii) $H^{\mathbf{X}}$ consists of an n-ary partial operation $h^{\mathbf{X}} \colon \operatorname{dom}(h^{\mathbf{X}}) \to X$ for each n-ary partial operation symbol $h \in H$, where $\operatorname{dom}(h^{\mathbf{X}}) \subseteq X^n$,

(iii) $R^{\mathbf{X}}$ consists of an n-ary relation $r^{\mathbf{X}} \subseteq X^n$ on X for each n-ary relation symbol $r \in R$,

(iv) $\langle X, \mathcal{T}^X\rangle$ is a topological space.

If G includes no nullary operations, then we will allow X to be empty. The **empty structure**, $\underset{\sim}{\varnothing}$, consists of the empty set with each operation, partial operation and relation being empty. In the actual cases that we will consider, the sets G, H and R will usually be either small or empty and the arities n will usually be small. Remember that the ultimate utility of natural dualities will hinge on our ability to construct a dual category $\mathcal{X}$ which is in a practical sense simpler than the corresponding original category $\mathcal{A}$, and this will normally mean that its type must be small.

Many of the familiar algebraic notions and results discussed in the previous section have more than one natural extension to structures with relations and partial operations. It is therefore important that we formulate explicitly the extensions of these notions that will be used in this book. Let

$$\mathbf{X} = \langle X; G^{\mathbf{X}}, H^{\mathbf{X}}, R^{\mathbf{X}}, \mathcal{T}^{\mathbf{X}}\rangle$$

be a structured topological space. If $h^{\mathbf{X}} \in G^{\mathbf{X}} \cup H^{\mathbf{X}}$ is n-ary, then the **graph** of $h^{\mathbf{X}}$ is the $(n+1)$-ary relation

$$\operatorname{graph}(h^{\mathbf{X}}) = \{(x_1, x_2, \dots, x_n, y) \in X^{n+1} \mid h^{\mathbf{X}}(x_1, x_2, \dots, x_n) = y\}$$

on X. If $g \in G^{\mathbf{X}}$ is nullary, then $\operatorname{graph}(g) = \{g\}$.

If g is a k-ary partial operation and $h_1, \dots, h_k$ are n-ary partial operations on X, then the **composition** $g(h_1, \dots, h_k)$ is the n-ary partial operation whose domain consists of those $x \in X^n$ for which the expression $g(h_1(x), \dots, h_k(x))$ is defined. Of course, $g(h_1(x), \dots, h_k(x))$ could be the **empty map**, $\eta \colon \varnothing \to M$, even when g and $h_1, \dots, h_n$ have non-empty domains. We shall allow both $k = 0$ and $n = 0$ in this definition, but some care is required. When $n = 0$, even though g might be partial, this construct produces either η or a (total) nullary operation: if $c_1, \dots, c_k$ are nullary with $(c_1, \dots, c_k) \in \operatorname{dom}(g)$, then we obtain the nullary operation $g(c_1,\dots,c_k)$. When $k=0$, the operation g is nullary and we declare that g composed with the empty set of n-ary total maps produces the constant total n-ary map with value g.

A set of partial operations on X is called an **enriched partial clone** on X if it includes the coordinate projections $\pi_i\colon X^n \to X$ for each $n \geqslant 1$ and is closed under composition. The **enriched partial clone generated** by a set P of partial operations on X is the smallest enriched partial clone on X containing P and is denoted by $[P]$. Note that, to guarantee closure under composition, $[P]$ may contain the empty map η. The **enriched partial clone of** $\mathbf{X} = \langle X; G^{\mathbf{X}}, H^{\mathbf{X}}, R^{\mathbf{X}}, \mathcal{T}^{\mathbf{X}}\rangle$ is defined to be $[G^{\mathbf{X}} \cup H^{\mathbf{X}}]$. Unlike clones on algebras, these clones are *enriched* to allow the possibility that they might include nullary operations as well.

Now consider another structured topological space

$$\mathbf{Y} = \langle Y; G^{\mathbf{Y}}, H^{\mathbf{Y}}, R^{\mathbf{Y}}, \mathcal{T}^{\mathbf{Y}}\rangle.$$

A continuous map $\varphi\colon X \to Y$ is a **morphism**, written $\varphi\colon \mathbf{X} \to \mathbf{Y}$, if it preserves each member of R and the graph of each member of $G \cup H$. Written out in detail, φ is a morphism if

(i) for each n-ary $g \in G$ and each $(x_1, x_2, \dots, x_n) \in \mathbf{X}^n$, we have

$$\varphi(g^{\mathbf{X}}(x_1, x_2, \dots, x_n)) = g^{\mathbf{Y}}(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_n)),$$

(ii) for each n-ary $h \in H$ and each $(x_1, x_2, \dots, x_n) \in \operatorname{dom}(h^{\mathbf{X}})$ we have

$$(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_n)) \in \operatorname{dom}(h^{\mathbf{Y}}) \text{ and}$$

$$\varphi(h^{\mathbf{X}}(x_1, x_2, \dots, x_n)) = h^{\mathbf{Y}}(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_n)),$$

(iii) for each n-ary $r \in R$ and each $(x_1, x_2, \dots, x_n) \in r^{\mathbf{X}}$ we have

$$(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_n)) \in r^{\mathbf{Y}}, \text{ and}$$

(iv) φ is continuous.

The class of all structured topological spaces of a given type $\langle G, H, R\rangle$ with these morphisms forms the broad category in which we will work. In the case that G contains no nullary operations and the empty structure is allowed, for each structure $\mathbf{X}$ of the type, there is a unique (empty) morphism from $\underset{\sim}{\varnothing}$ to $\mathbf{X}$ and there are no morphisms in the other direction when $\mathbf{X}$ is non-empty.

The structure $\mathbf{Y} = \langle Y; G^{\mathbf{Y}}, H^{\mathbf{Y}}, R^{\mathbf{Y}}, \mathcal{T}^{\mathbf{Y}}\rangle$ is called a **substructure** of the structured topological space $\mathbf{X} = \langle X; G^{\mathbf{X}}, H^{\mathbf{X}}, R^{\mathbf{X}}, \mathcal{T}^{\mathbf{X}}\rangle$, written $\mathbf{Y} \leqslant \mathbf{X}$, provided that $Y \subseteq X$ and

(i) for each n-ary $g \in G$ the operations $g^{\mathbf{Y}}$ and $g^{\mathbf{X}}$ agree on Y^n,

(ii) for each n-ary $h \in H$, we have $\operatorname{dom}(h^{\mathbf{Y}}) = \operatorname{dom}(h^{\mathbf{X}}) \cap Y^n$, and $h^{\mathbf{Y}}$ agrees with $h^{\mathbf{X}}$ on this set,

(iii) for each n-ary $r \in R$, we have $r^{\mathbf{Y}} = r^{\mathbf{X}} \cap Y^n$, and

(iv) $\mathcal{T}^{\mathbf{Y}}$ is the relative topology obtained from $\mathcal{T}^{\mathbf{X}}$.

Notice that the empty structure, $\underset{\sim}{\varnothing}$, is a substructure of $\mathbf{X}$ exactly when G includes no nullary operation symbols. Consequently the substructures of $\mathbf{X}$ are always closed under intersection and therefore form a complete lattice.

Products of structured topological spaces of the same type are defined over *non-empty index sets* in the usual manner, using pointwise total operations and relations and the product topology, and partial operations are defined pointwise whenever they are defined at each point. Concrete products defined in this way are products in the category of structured topological spaces in the category-theoretic sense (Exercise 1.12).

A morphism $\varphi\colon \mathbf{X} \to \mathbf{Y}$ is an **embedding** of $\mathbf{X}$ into $\mathbf{Y}$ if the image $\varphi(\mathbf{X})$ forms a substructure of $\mathbf{Y}$ and φ is an isomorphism from $\mathbf{X}$ onto $\varphi(\mathbf{X})$. While this definition requires verification of many details, a number of them follow automatically in the context where we will use it.

From this point onward we will relax our use of the superscripts on $G^{\mathbf{X}}$, $H^{\mathbf{X}}$ and $R^{\mathbf{X}}$ and their members, using them only when there may be a danger of ambiguity.

Let $\underset{\sim}{\mathbf{M}} := \langle M; G, H, R, \mathcal{T}\rangle$ be a structured topological space where M is finite and the topology $\mathcal{T}$ is discrete. For a class $\mathcal{Y}$ of structured topological spaces of the same type, we define $\mathbb{I}(\mathcal{Y})$, $\mathbb{S}_c(\mathcal{Y})$, $\mathbb{P}^+(\mathcal{Y})$, respectively, to be the class of isomorphic copies, topologically closed substructures and direct products (over non-empty index sets) of members of $\mathcal{Y}$. Beginning with $\underset{\sim}{\mathbf{M}}$ we generate the class

$$\mathcal{X} := \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$$

consisting of all isomorphic copies of closed substructures of direct powers of copies of $\underset{\sim}{\mathbf{M}}$. Because $\underset{\sim}{\mathbf{M}}$ is finite and discrete its topology is Boolean. As a result the topology on each other member of $\mathcal{X}$ is also Boolean (Lemma B.5). Thus $\mathcal{X}$ forms a category of **structured Boolean spaces**.

The dual of a finitely generated quasi-variety $\mathcal{A} = \mathbb{ISP}(\underline{\mathbf{M}})$ under a natural duality will always be a category of the form $\mathcal{X} = \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ where $\underset{\sim}{\mathbf{M}}$ is a finite discrete structure. The next section and the next two chapters will focus on the problem of determining $\underset{\sim}{\mathbf{M}}$ when we are given $\underline{\mathbf{M}}$.

Predualities We are now ready to begin assembling the pieces. In this section we will present a uniform method whereby, starting with an arbitrary finite algebra $\underline{\mathbf{M}}$, we can always construct many different finite structures $\underset{\sim}{\mathbf{M}}$ so that $\mathcal{A} = \mathbb{ISP}(\underline{\mathbf{M}})$ is dually adjoint to $\mathcal{X} = \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$. From this vast array of dual adjunctions we can hope to find many nice dual equivalences. As we proceed we will rely on the model of distributive lattices and bounded Priestley spaces to guide us in defining the structure $\underset{\sim}{\mathbf{M}}$, the contravariant functors D and E, and the special morphisms $e_{\mathbf{A}}$ and $\varepsilon_{\mathbf{X}}$ in a general setting. At the end of this section two questions will remain: for which choices of $\underset{\sim}{\mathbf{M}}$ is $\langle D, E, e, \varepsilon\rangle$ a dual representation and for which choices is it a dual equivalence? Answering these questions will be the goals of Chapters 2 and 3, respectively.

We begin with a fixed finite algebra $\underline{\mathbf{M}}$ generating a quasi-variety $\mathcal{A} = \mathbb{ISP}(\underline{\mathbf{M}})$. The example of distributive lattices and bounded Priestley spaces suggests that $\underset{\sim}{\mathbf{M}}$ be a structure sharing the same underlying set M as $\underline{\mathbf{M}}$. Loosely speaking, we think of M as a **schizophrenic object** which has an algebraic personality $\underline{\mathbf{M}}$ and a topological personality $\underset{\sim}{\mathbf{M}}$. In its two personalities it generates two superficially different categories $\mathcal{A}$ and $\mathcal{X}$, but under the right conditions these two categories prove to be mirror images of one another.

How do we define the contravariant functors D and E in this setting? We try to 'define'

$$D(\mathbf{A}) := \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}}) \subseteq M^A$$

in general. Viewing M as a discrete space, we notice that $\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ is always a closed subspace of the direct power M^A and is therefore a Boolean space (Exercise 1.19). It then remains to impose a structure on $\underset{\sim}{\mathbf{M}}$ that assures us that the homset $\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ is always a *substructure* of $\underset{\sim}{\mathbf{M}}^A$.

Before doing this, we consider the definition of E. We 'define'

$$E(\mathbf{X}) := \mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}}) \subseteq M^X$$

in general. Once again, we need to impose conditions on $\underset{\sim}{\mathbf{M}}$ which ensure that the homset $\mathcal{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$ is a subalgebra of the direct power $\underline{\mathbf{M}}^X$.

Given a type of algebras there is, up to isomorphism, a unique one-element algebra of that type. If we consider a type of structures with partial operations or relations, we notice that there are many non-isomorphic one-element structures, each determined by taking different partial operations and relations to be non-empty. We will refer to a one-element structure in which *all* partial operations and relations are non-empty as the **complete one-element structure** of that type.

The first and second conditions of the following theorem are the two conditions that we are seeking. These are together equivalent to the third condition, which is stated from the point of view of the structure on $\underline{\mathbf{M}}$, and the fourth condition, which is stated from the point of view of the structure on $\underset{\sim}{\mathbf{M}}$. Notice that in the fourth condition, if $n > 0$, then the graph of an n-ary (partial) operation h is a subalgebra of $\underline{\mathbf{M}}^{n+1}$ if and only if h is a homomorphism from a subalgebra of $\underline{\mathbf{M}}^n$ into $\underline{\mathbf{M}}$ (Exercise 1.20). Condition (iv) gives us a simple method of constructing *many* choices of $\underset{\sim}{\mathbf{M}}$ for any given algebra $\underline{\mathbf{M}}$.

1.5.2 Preduality Theorem. *Let $\underline{\mathbf{M}}$ be a finite algebra and let $\underset{\sim}{\mathbf{M}}$ be a discrete topological structure such that each relation and the domain of each partial operation of $\underset{\sim}{\mathbf{M}}$ are non-empty and $\underset{\sim}{\mathbf{M}}$ has the same underlying set M as $\underline{\mathbf{M}}$. Then conditions* (ii), (iii) *and* (iv) *below are equivalent and imply condition* (i)*:*

(i) $\mathcal{A}(\mathbf{A},\underline{\mathbf{M}})$ *is a closed substructure of* $\underset{\sim}{\mathbf{M}}^A$*, for each* $\mathbf{A}\in\mathcal{A}$*;*
(ii) $\mathcal{X}(\mathbf{X},\underset{\sim}{\mathbf{M}})$ *is a subalgebra of* $\underline{\mathbf{M}}^X$*, for each* $\mathbf{X}\in\mathcal{X}$*;*
(iii) *each non-nullary operation of* $\underline{\mathbf{M}}$ *is a morphism from a power of* $\underset{\sim}{\mathbf{M}}$ *into* $\underset{\sim}{\mathbf{M}}$*, and each distinguished element of* $\underline{\mathbf{M}}$ *forms a complete one-element substructure of* $\underset{\sim}{\mathbf{M}}$*;*
(iv) *each relation, the graph of each non-nullary operation and the graph of each partial operation of* $\underset{\sim}{\mathbf{M}}$ *are subalgebras of powers of* $\underline{\mathbf{M}}$*, and each distinguished element of* $\underset{\sim}{\mathbf{M}}$ *is a one-element subalgebra of* $\underline{\mathbf{M}}$*.*

An n-ary relation on M is called **algebraic over** $\underline{\mathbf{M}}$ if it forms a subalgebra of $\underline{\mathbf{M}}^n$. An n-ary operation g on M is called **algebraic over** $\underline{\mathbf{M}}$ if it is a homomorphism from $\underline{\mathbf{M}}^n$ to $\underline{\mathbf{M}}$. In particular, a nullary operation is algebraic over $\underline{\mathbf{M}}$ if the element it distinguishes forms a one-element subalgebra of $\underline{\mathbf{M}}$. An n-ary partial operation h on M is called **algebraic over** $\underline{\mathbf{M}}$ if it is a homomorphism from a subalgebra of $\underline{\mathbf{M}}^n$ to $\underline{\mathbf{M}}$. By Exercise 1.20, this is equivalent to saying that the graphs of g and h form subalgebras of $\underline{\mathbf{M}}^{n+1}$. In this language, condition (iv) of the Preduality Theorem 1.5.2 says precisely that each of the operations, partial operations and relations of $\underset{\sim}{\mathbf{M}}$ is algebraic over $\underline{\mathbf{M}}$. When this occurs, we say that $\underset{\sim}{\mathbf{M}}$ is **algebraic over** $\underline{\mathbf{M}}$ and that $\underset{\sim}{\mathbf{M}}$ **yields a preduality on** $\mathcal{A}$.

Under a preduality the schizophrenia exhibited by the pair $\underline{\mathbf{M}}$–$\underset{\sim}{\mathbf{M}}$ proves to be contagious. We have sets which bear the identity of objects in one category but behave as relations in the other, or as morphisms in one category but as operations in the other. We will normally adopt notation appropriate to the identity under which the entity in question arises, while retaining the flexibility to shift to the other point of view whenever necessary. Many startling punches in duality theory result from such well-timed shifts.

Once $\underset{\sim}{\mathbf{M}}$ is algebraic over $\underline{\mathbf{M}}$, conditions (i) and (ii) of the Preduality Theorem 1.5.2 allow us to define $D\colon \mathcal{A}\to\mathcal{X}$ and $E\colon \mathcal{X}\to\mathcal{A}$ by

$$D(\mathbf{A}) = \mathcal{A}(\mathbf{A},\underline{\mathbf{M}}) \leqslant \underset{\sim}{\mathbf{M}}^A \quad\text{and}\quad E(\mathbf{X}) = \mathcal{X}(\mathbf{X},\underset{\sim}{\mathbf{M}}) \leqslant \underline{\mathbf{M}}^X.$$

Moreover, it turns out that there are simple and natural candidates for the remaining components of a dual adjunction in this setting. To complete the definitions of the contravariant functors D and E, consider $\mathbf{A},\mathbf{B}\in\mathcal{A}$ and a homomorphism $u\colon \mathbf{A}\to\mathbf{B}$. We define

$$D(u)\colon D(\mathbf{B})\to D(\mathbf{A}) \quad\text{by}\quad D(u)(x) = x\circ u.$$

Similarly, for $\mathbf{X},\mathbf{Y}\in\mathcal{X}$ and $\varphi\colon \mathbf{X}\to\mathbf{Y}$, we define

$$E(\varphi)\colon E(\mathbf{Y})\to E(\mathbf{X}) \quad\text{by}\quad E(\varphi)(\alpha) = \alpha\circ\varphi.$$

For each $\mathbf{A}\in\mathcal{A}$ and each $\mathbf{X}\in\mathcal{X}$ we define the **evaluation maps**

$$e_{\mathbf{A}}\colon \mathbf{A}\to ED(\mathbf{A}) \quad\text{by}\quad e_{\mathbf{A}}(a)(x) = x(a)$$

$$\begin{tikzcd} \mathbf{A} \arrow[r, "u"] \arrow[d, "e_{\mathbf{A}}"'] & \mathbf{B} \arrow[d, "e_{\mathbf{B}}"] \\ ED(\mathbf{A}) \arrow[r, "ED(u)"'] & ED(\mathbf{B}) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{X} \arrow[r, "\varphi"] \arrow[d, "\varepsilon_{\mathbf{X}}"'] & \mathbf{Y} \arrow[d, "\varepsilon_{\mathbf{Y}}"] \\ DE(\mathbf{X}) \arrow[r, "DE(\varphi)"'] & DE(\mathbf{Y}) \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{A} \arrow[r, "e_{\mathbf{A}}"] \arrow[dr, "u"'] & ED(\mathbf{A}) \arrow[d, "E(\varphi)"] \\ & E(\mathbf{X}) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{X} \arrow[r, "\varepsilon_{\mathbf{X}}"] \arrow[dr, "\varphi"'] & DE(\mathbf{X}) \arrow[d, "D(u)"] \\ & D(\mathbf{A}) \end{tikzcd}$$

FIGURE 1.2 a dual adjunction

and

$$\varepsilon_{\mathbf{X}} \colon \mathbf{X} \to DE(\mathbf{X}) \quad \text{by} \quad \varepsilon_{\mathbf{X}}(x)(\alpha) = \alpha(x).$$

We say that $\langle D, E, e, \varepsilon \rangle$ is a **dual adjunction** between $\mathcal{A}$ and $\mathcal{X}$, that D and E are **dually adjoint**, that E is a **dual adjoint to** D, and that D is a **dual adjoint to** E if the following conditions hold:

(i) for $u \colon \mathbf{A} \to \mathbf{B}$ in $\mathcal{A}$ and $\varphi \colon \mathbf{X} \to \mathbf{Y}$ in $\mathcal{X}$, the two squares in Figure 1.2 **commute**, that is, $e_{\mathbf{B}} \circ u = ED(u) \circ e_{\mathbf{A}}$ and $\varepsilon_{\mathbf{Y}} \circ \varphi = DE(\varphi) \circ \varepsilon_{\mathbf{X}}$,

(ii) for $\mathbf{A} \in \mathcal{A}$ and $\mathbf{X} \in \mathcal{X}$ there is a bijection between $\mathcal{A}(\mathbf{A}, E(\mathbf{X}))$ and $\mathcal{X}(\mathbf{X}, D(\mathbf{A}))$ associating u and φ as given in the commuting triangles of Figure 1.2, that is, $u = E(D(u) \circ \varepsilon_{\mathbf{X}}) \circ e_{\mathbf{A}}$ and $\varphi = D(E(\varphi) \circ e_{\mathbf{A}}) \circ \varepsilon_{\mathbf{X}}$.

Those already experienced with categories will notice that $\mathrm{id}_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{A}$ and $ED \colon \mathcal{A} \to \mathcal{A}$ are covariant functors and that the left square of Figure 1.2 says precisely that $e \colon \mathrm{id}_{\mathcal{A}} \to ED$ is a **natural transformation**. A similar observation can be made about the right square.

The next theorem shows that in this context a large part of the construction we seek is present as soon as $\underset{\sim}{\mathbf{M}}$ is algebraic over $\underline{\mathbf{M}}$.

1.5.3 Dual Adjunction Theorem. *If $\underset{\sim}{\mathbf{M}}$ is algebraic over $\underline{\mathbf{M}}$, then $\langle D, E, e, \varepsilon \rangle$ is a dual adjunction between $\mathcal{A}$ and $\mathcal{X}$ such that the evaluation maps $e_{\mathbf{A}}$ and $\varepsilon_{\mathbf{X}}$ are embeddings.*

We *assume without further mention* that all operations, partial operations and relations on $\underset{\sim}{\mathbf{M}}$ are algebraic over $\underline{\mathbf{M}}$.

Update 1

The restriction to *finite algebras* in the choice of $\underline{\mathbf{M}}$ can be and has been relaxed.

Infinite rather than finite. In the original 1980 Davey–Werner paper [40], infinite algebras $\underline{\mathbf{M}}$ with a compatible compact topology were allowed. This brings Pontryagin duality for abelian groups under the natural-duality umbrella. As this forces us into the realm of topological algebra and there is a paucity of natural examples, this direction has been little pursued. Nevertheless, there are several papers that do allow for an infinite $\underline{\mathbf{M}}$ and yield a range of examples: see [41, 42, 35, 29, 23].

Structures rather than algebras. The lack of symmetry that results from restricting $\underline{\mathbf{M}}$ to be an algebra while allowing $\underset{\sim}{\mathbf{M}}$ to be a structure can be removed by allowing both $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}$ to be structures. Much, but not all of the theory goes over to this setting: see [47, 17] for the theory, [33] for an approach via a natural Galois connection on partial operations, and [54, 55, 56, 57] for examples.

Empty structures. While we include the empty structure in the category $\mathcal{X}$, we have followed the usual algebraic convention of excluding the empty algebra from the category $\mathcal{A}$, but this is not necessary. Indeed, the decision to include or exclude both empty and one-element structures in the category $\mathcal{A}$ is one of personal preference, and all variants can be accommodated. For a discussion of the four different settings that arise we refer to the appendix of [33].

Chapter 2: Natural Dualities

Having set the scene in the previous chapter, we can now begin to addess the most immediate issues. As algebraists, our first aim is to obtain a representation of each algebra in $\mathcal{A} = \mathbb{ISP}(\underline{\mathbf{M}})$ as an algebra of continuous, structure-preserving maps.

Given a discrete topological structure $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ which is algebraic over $\underline{\mathbf{M}}$, we have a dual adjunction $\langle D, E, e, \varepsilon\rangle$ between $\mathcal{A}$ and $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ as described in the last section of the previous chapter. In particular, by the Dual Adjunction Theorem 1.5.3, the homomorphism $e_{\mathbf{A}} \colon \mathbf{A} \to ED(\mathbf{A})$ is an embedding for all $\mathbf{A} \in \mathcal{A}$. We shall say that $\underset{\sim}{\mathbf{M}}$ **yields a (natural) duality** on an algebra $\mathbf{A}$ in $\mathcal{A}$ if $e_{\mathbf{A}}$ is an isomorphism, that is, if the only continuous structure-preserving maps from $D(\mathbf{A})$ to $\underset{\sim}{\mathbf{M}}$ are the evaluations. If $\mathcal{C} \subseteq \mathcal{A}$ and $\underset{\sim}{\mathbf{M}}$ yields a duality on every algebra $\mathbf{A}$ in $\mathcal{C}$, then we say that $\underset{\sim}{\mathbf{M}}$ **yields a (natural) duality on** $\mathcal{C}$. Thus $\underset{\sim}{\mathbf{M}}$ yields a natural duality on $\mathcal{A}$ precisely when the preduality determined by $\underset{\sim}{\mathbf{M}}$ is a dual representation. Instead of saying that $\underset{\sim}{\mathbf{M}}$ yields a duality, we shall sometimes say that $G \cup H \cup R$ **yields a duality**. Putting the emphasis back on the algebra $\underline{\mathbf{M}}$, we say that $\underline{\mathbf{M}}$ **admits a (natural) duality**, or that $\underline{\mathbf{M}}$ is **dualisable**, if there exists some structure $\utilde{\mathbf{M}}$ which yields a duality on $\mathcal{A}$, in which case we often say simply that $\utilde{\mathbf{M}}$ (or $G \cup H \cup R$) **dualises** $\underline{\mathbf{M}}$.

Brute force We shall see in the next chapter that the operations in G and the partial operations in H play a vital role when we wish to upgrade a natural duality (= dual representation) to a full duality (= dual equivalence). Nevertheless, our first lemma implies that if we are trying to prove only that $\utilde{\mathbf{M}}$ yields a duality on $\mathcal{A}$, we may delete an operation from G or a partial operation from H provided we add its graph to the set R of relations. Let $\mathrm{dom}(h) \subseteq M^n$ and $h\colon \mathrm{dom}(h) \to M$. By Exercise 1.20, $\mathrm{graph}(h)$ is a subalgebra of $\underline{\mathbf{M}}^{n+1}$ if and only if $\mathrm{dom}(h)$ is a subalgebra of $\underline{\mathbf{M}}^n$ and $h\colon \mathrm{dom}(h) \to \underline{\mathbf{M}}$ is a homomorphism, that is, h is algebraic over $\underline{\mathbf{M}}$ if and only if $\mathrm{graph}(h)$ is algebraic over $\underline{\mathbf{M}}$. Thus it makes sense to delete h from G or H and to add $\mathrm{graph}(h)$ to R.

2.1.2 Lemma. *Let $\utilde{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$, define $R' := R \cup \{\mathrm{graph}(h) \mid h \in G \cup H\}$ and let $\utilde{\mathbf{M}}' = \langle M; R', \mathcal{T}\rangle$. Then $\utilde{\mathbf{M}}$ yields a duality on $\mathbf{A} \in \mathcal{A}$ if and only if $\utilde{\mathbf{M}}'$ yields a duality on $\mathbf{A}$.*

If $\utilde{\mathbf{M}} = \langle M; R, \mathcal{T}\rangle$ yields a duality on $\mathbf{A}$, for some set R of algebraic relations, then, since the evaluation maps $e_{\mathbf{A}}(a)\colon D(\mathbf{A}) \to M$ preserve every algebraic relation on $\underline{\mathbf{M}}$, for any set R' of algebraic relations which contains R, the structure $\utilde{\mathbf{M}}' = \langle M; R', \mathcal{T}\rangle$ will also yield a duality on $\mathbf{A}$. In particular, $\utilde{\mathbf{M}} = \langle M; \mathcal{B}, \mathcal{T}\rangle$ will yield a duality on $\mathbf{A}$ where

$$\mathcal{B} = \bigcup \{\, \mathbb{S}(\underline{\mathbf{M}}^n) \mid n \geqslant 1 \,\}$$

is the set of *all* finitary algebraic relations on $\underline{\mathbf{M}}$. We refer to this as the **brute force** construction. Thus the issue of the existence of a duality may, on one level, be summed up as in the lemma below.

2.1.3 Lemma. *Let $\mathbf{A} \in \mathcal{A}$. The following are equivalent:*

(i) *there is some structure $\utilde{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ which yields a duality on $\mathbf{A}$;*
(ii) *there is some purely relational structure $\utilde{\mathbf{M}} = \langle M; R, \mathcal{T}\rangle$ which yields a duality on $\mathbf{A}$;*
(iii) *brute force yields a duality on $\mathbf{A}$ (that is, $\utilde{\mathbf{M}} = \langle M; \mathcal{B}, \mathcal{T}\rangle$ yields a duality on $\mathbf{A}$);*
(iv) *the evaluation maps $e_{\mathbf{A}}(a)\colon \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}}) \to M$, where $a \in A$, are the only continuous maps from $\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ to M which preserve every finitary algebraic relation on $\underline{\mathbf{M}}$.*

The brute force construction is an important theoretical tool, but in practice we try to make the structure on $\utilde{\mathbf{M}}$ as simple as possible. Part of the beauty of Priestley's duality for distributive lattices is that it is given by a single, particularly simple relation and two constants. It is highly unlikely that anyone would use the brute force duality for the class $\mathcal{D}$ of distributive lattices. Nevertheless, brute force does yield a duality on $\mathcal{D}$.

Duality theorems. We return now to the situation where $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ and we seek readily verifiable and preferably finitary conditions which will guarantee that $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$. The idea behind the First Duality Theorem is quite simple: since every algebra in $\mathcal{A}$ is a homomorphic image of a free algebra in $\mathcal{A}$ (Theorem A.2), in order to prove that $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$ it suffices to

- show that $\underset{\sim}{\mathbf{M}}$ yields a duality on all the free algebras in $\mathcal{A}$,
  and then
- show that the class of all algebras on which $\underset{\sim}{\mathbf{M}}$ yields a duality is closed under homomorphic images.

Obtaining necessary and sufficient finitary conditions for $\underset{\sim}{\mathbf{M}}$ to yield a duality on all free algebras in $\mathcal{A}$ is quite straightforward. We refer the reader to Appendix A for a review of free algebras.

2.2.1 Lemma. *Let S be a non-empty set, let $\mathbf{F}_{\underline{\mathbf{M}}}(S)$ be the free algebra of S-ary term functions over $\underline{\mathbf{M}}$ and let*

$$\rho_S \colon D(\mathbf{F}_{\underline{\mathbf{M}}}(S)) = \mathcal{A}(\mathbf{F}_{\underline{\mathbf{M}}}(S), \underline{\mathbf{M}}) \to \underset{\sim}{\mathbf{M}}^S$$

be the map which restricts each homomorphism $x \colon \mathbf{F}_{\underline{\mathbf{M}}}(S) \to \underline{\mathbf{M}}$ to the generators, that is, $\rho_S(x)(s) = x(\pi_s)$. Then ρ_S is an isomorphism in $\mathcal{X}$.

The structure $\underset{\sim}{\mathbf{M}}$ is **injective** in the category $\mathcal{X}$ if, for every morphism $\alpha \colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ and embedding $\varphi \colon \mathbf{X} \to \mathbf{Y}$ in $\mathcal{X}$, there is a morphism $\beta \colon \mathbf{Y} \to \underset{\sim}{\mathbf{M}}$ such that $\beta \circ \varphi = \alpha$. The injectivity of $\underset{\sim}{\mathbf{M}}$ will play a central role both here and in Chapter 3. Injectivity of $\underline{\mathbf{M}}$ in $\mathcal{A}$ will also be important in Chapter 3 but will not be considered here.

2.2.2 First Duality Theorem. *The following are equivalent:*

(i) *$\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$;*

(ii) *for all $\mathbf{A} \in \mathcal{A}$, every morphism $\alpha \colon D(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$ extends to an A-ary term function $t \colon M^A \to M$;*

(iii) *the following two conditions hold—*

(INJ) *$\underset{\sim}{\mathbf{M}}$ is injective with respect to those embeddings in $\mathcal{X}$ which are of the form $D(u) \colon D(\mathbf{A}) \to D(\mathbf{B})$ where $u \colon \mathbf{B} \to \mathbf{A}$ is a surjective homomorphism, that is, for each morphism $\alpha \colon D(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$ there exists a morphism $\beta \colon D(\mathbf{B}) \to \underset{\sim}{\mathbf{M}}$ such that $\beta \circ D(u) = \alpha$,*

(CLO) *for each $n \in \mathbb{N}$, every morphism $t \colon \underset{\sim}{\mathbf{M}}^n \to \underset{\sim}{\mathbf{M}}$ is an n-ary term function on $\underline{\mathbf{M}}$.*

We begin by proving that (CLO) does indeed capture the fact that $\underset{\sim}{\mathbf{M}}$ yields a duality on the free algebras in $\mathcal{A}$.

2.2.3 Proposition.

(i) *For a fixed set S, the structure $\underset{\sim}{\mathbf{M}}$ yields a duality on the free algebra $\mathbf{F}_{\underline{\mathbf{M}}}(S)$ if and only if the following condition holds:*

$(\mathrm{CLO})_S$ *every morphism $t\colon \underset{\sim}{\mathbf{M}}^S \to \underset{\sim}{\mathbf{M}}$ is an S-ary term function on $\underline{\mathbf{M}}$.*

(ii) *The following are equivalent:*

- (a) (CLO) *holds;*
- (b) $(\mathrm{CLO})_S$ *holds for every non-empty set S;*
- (c) $\underset{\sim}{\mathbf{M}}$ *yields a duality on the finitely generated free algebras in $\mathcal{A}$;*
- (d) $\underset{\sim}{\mathbf{M}}$ *yields a duality on all free algebras in $\mathcal{A}$.*

As an immediate consequence of Lemma 2.2.1 and Proposition 2.2.3 we obtain the following corollary, which says that powers of $\underset{\sim}{\mathbf{M}}$ are dual to free algebras in $\mathcal{A}$ under very weak circumstances.

2.2.4 Corollary. *If S is a non-empty set and $\underset{\sim}{\mathbf{M}}$ is algebraic over $\underline{\mathbf{M}}$, then*

(i) $D(\mathbf{F}_{\underline{\mathbf{M}}}(S)) \cong \underset{\sim}{\mathbf{M}}^S$ *and*

(ii) $E(\underset{\sim}{\mathbf{M}}^S) = \mathbf{F}_{\underline{\mathbf{M}}}(S)$ *if* (CLO) *holds.*

By the Preduality Theorem 1.5.2, every term function of $\underline{\mathbf{M}}$ is a morphism in $\mathcal{X}$. The condition (CLO) adds the converse: duality requires that the (finitary) term functions on $\underline{\mathbf{M}}$ must be exactly the morphisms from finite powers of $\underset{\sim}{\mathbf{M}}$ into $\underset{\sim}{\mathbf{M}}$. Thus (CLO) says precisely that $G \cup H \cup R$ determines the clone of term functions on $\underline{\mathbf{M}}$. It is important to note that (CLO) is not sufficient to guarantee duality. There is a two-element set R of relations which determines the clone of the three-element Kleene algebra but does not yield a duality. Nevertheless, the addition of one further relation to R does yield a duality. (See (CLO) versus (IC) in 4.3.12.)

The condition (INJ) guarantees that if $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathbf{B} \in \mathcal{A}$ then $\underset{\sim}{\mathbf{M}}$ yields a duality on every homomorphic image of $\mathbf{B}$ in $\mathcal{A}$ (Exercise 2.2). Thus the conditions (CLO) and (INJ) lead us along a natural algebraic path: we first show that $\underset{\sim}{\mathbf{M}}$ yields a duality on the free algebras by proving (CLO) and then, since every algebra is a homomorphic image of a free algebra, show that the duality extends to arbitrary algebras by proving (INJ).

In order to prove (INJ), we will begin by verifying a more accessible special case of the injectivity of $\underset{\sim}{\mathbf{M}}$ in $\mathcal{X}$. The condition (CLO) together with the assertion that $\underset{\sim}{\mathbf{M}}$ is injective in $\mathcal{X}_{\mathrm{fin}}$ are equivalent to the following simple **interpolation condition**:

(IC) *for each $n \in \mathbb{N}$ and each substructure $\mathbf{X}$ of $\underset{\sim}{\mathbf{M}}^n$, every morphism $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ extends to a term function $t\colon M^n \to M$ of the algebra $\underline{\mathbf{M}}$.*

They are also equivalent to a statement about duality.

2.2.5 Lemma. *The following are equivalent:*

(i) (IC) *holds;*

(ii) (CLO) *holds and* $\underset{\sim}{\mathbf{M}}$ *is injective in* $\mathcal{X}_{\mathrm{fin}}$*;*

(iii) $\underset{\sim}{\mathbf{M}}$ *yields a duality on* $\mathcal{A}_{\mathrm{fin}}$ *and is injective in* $\mathcal{X}_{\mathrm{fin}}$.

We would like to obtain a duality for $\mathcal{A}$ in two steps: first show that $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}_{\mathrm{fin}}$, for example by verifying that (IC) holds, and then apply some general theory to show that the duality lifts automatically to a duality on the whole of $\mathcal{A}$. Our next two theorems shows that this is achievable provided $\underset{\sim}{\mathbf{M}}$ enjoys some degree of finiteness.

If $\underset{\sim}{\mathbf{M}} = \langle M; G, R, \mathcal{T}\rangle$, that is, every operation of $\underset{\sim}{\mathbf{M}}$ is total rather than partial, then we call $\underset{\sim}{\mathbf{M}}$ a **total structure**.

2.2.7 Second Duality Theorem. *Assume that* $\underset{\sim}{\mathbf{M}}$ *is a total structure with* R *finite. If* (IC) *holds, then* $\underset{\sim}{\mathbf{M}}$ *yields a duality on* $\mathcal{A}$ *and is injective in* $\mathcal{X}$.

This result is rather surprising. It gives us simple finitary conditions which yield both a dual adjunction between the categories $\mathcal{A}$ and $\mathcal{X}$ and a topological representation of each algebra in $\mathcal{A}$, but it requires us to do no category theory and no topology! While (IC) is sufficient, under certain circumstances, to show that $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$, it is not necessary (see Exercise 4.11 and Lemma 7.8.6).

The final theorem in this section gives a reduction to the finite case without establishing (IC) and therefore yields no information on the injectivity of $\underset{\sim}{\mathbf{M}}$ in $\mathcal{X}$. This important result was proved by R. Willard (private communication)[2] and independently by L. Zádori [81]. Our proof is a modified version of Willard's proof. Zádori's proof is given in Chapter 10 (see 10.6.4). If $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ and G, H and R are finite, then we say that $\underset{\sim}{\mathbf{M}}$ is of **finite type**.

2.2.11 Duality Compactness Theorem. *If* $\underset{\sim}{\mathbf{M}}$ *yields a duality on* $\mathcal{A}_{\mathrm{fin}}$ *and is of finite type, then* $\underset{\sim}{\mathbf{M}}$ *yields a duality on* $\mathcal{A}$.

Combined with Lemma 2.2.5, the Duality Compactness Theorem produces a variant of the Second Duality Theorem.

2.2.12 Corollary. *Assume that* $\underset{\sim}{\mathbf{M}}$ *is of finite type. If* (IC) *holds, then* $\underset{\sim}{\mathbf{M}}$ *yields a duality on* $\mathcal{A}$ *and is injective in* $\mathcal{X}_{\mathrm{fin}}$.

In every known example of a natural duality we have been able to use a structure $\underset{\sim}{\mathbf{M}}$ of finite type. Hence, we close this section by posing a fundamental question.

2.2.13 Finite Type Problem. [3] *Is there a finite algebra* $\underline{\mathbf{M}}$ *for which some choice of* $\underset{\sim}{\mathbf{M}}$ (*and therefore* $\underset{\sim}{\mathbf{M}} = \langle M; \mathcal{B}, \mathcal{T}\rangle$) *yields a duality on* $\mathcal{A}$ *but no choice of* $\underset{\sim}{\mathbf{M}}$ *of finite type yields a duality on* $\mathcal{A}$*?*

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[2] Willard proved this important result in the early 1990s, but published his proof in 1999 [80].

[3] This problem is solved in [64], where it is proved that the answer is no for unary algebras, and in [65], where it is shown that the answer is yes for non-unary algebras.

Taming brute force with near-unanimity. In the light of the Second Duality Theorem, the natural question now is, *'How can we force* (IC) *to hold?'* The answer is, *'Use brute force!'*

2.3.1 Brute Force Duality Theorem. *Brute force yields a duality on $\mathcal{A}_{\text{fin}}$. Indeed, if $\underset{\sim}{\mathbf{M}} = \langle M; \mathcal{B}, \mathcal{T}\rangle$, then* (IC) *holds and therefore $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}_{\text{fin}}$ and $\underset{\sim}{\mathbf{M}}$ is injective in $\mathcal{X}_{\text{fin}}$.*

Even though brute force always yields a duality on the finite algebras in $\mathcal{A}$, it does not always yield a duality on $\mathcal{A}$. We shall see in Chapter 10 that not even brute force will yield a duality on the variety of implication algebras $\mathcal{A} := \mathbb{ISP}(\underline{\mathbf{I}})$, where $\underline{\mathbf{I}} := \langle \{0,1\}; \rightarrow, 1\rangle$ and $\rightarrow$ is the usual Boolean implication. If we are willing to use a proper *class* of possibly *infinitary* relations, we can obtain a duality for $\mathcal{A}$ without the use of any topology. Let $\mathcal{B}_\infty$ denote the class consisting of all subalgebras of $\underline{\mathbf{M}}^\kappa$ where κ ranges over all non-zero cardinals. Then a simple modification of the proof of 2.3.1 shows that $\underset{\sim}{\mathbf{M}}_\infty := \langle M; \mathcal{B}_\infty\rangle$ yields a duality on $\mathcal{A}$. This is the **ultimate brute force** construction. (See also Exercise 2.5, where we check that the omission of topology from $\underset{\sim}{\mathbf{M}}_\infty$ is not a typo!)

While brute force may not always yield a duality on $\mathcal{A}$, 2.2.6, 2.2.8 and 2.2.14 combine to show that brute force always yields a duality on the free algebras in $\mathcal{A}$. This gives a standard result of clone theory as an immediate corollary.

2.3.2 Corollary. *If $\underline{\mathbf{M}}$ is a finite algebra, then the set $\mathcal{B}$ of all finitary algebraic relations on $\underline{\mathbf{M}}$ determines the clone of $\underline{\mathbf{M}}$, that is, for all $n \in \mathbb{N}$, a map $t\colon M^n \to M$ is a term function of $\underline{\mathbf{M}}$ if and only if it preserves all finitary algebraic relations on $\underline{\mathbf{M}}$.*

The Second Duality Theorem and the Brute Force Duality Theorem are in a tug-of-war. The former says that (IC) will give us duality if we use only finitely many (partial) operations and relations. The latter tells us that we get (IC) if we are willing to put in all possible relations. In order to bridge the gap, we need a condition which will ensure that preservation of certain finite sets of relations guarantees preservation of all algebraic relations. Such a condition was discovered by K. Baker and A. Pixley [3], quite independently of duality theory. They call a $(k+1)$-ary term $n(x_1, \dots, x_{k+1})$, with k at least 2, a **near-unanimity term** on $\underline{\mathbf{M}}$ if $\underline{\mathbf{M}}$ satisfies the identities

$$n(x, \dots, x, y) \approx n(x, \dots, x, y, x) \approx \cdots \approx n(y, x, \dots, x) \approx x.$$

A ternary near-unanimity term on $\underline{\mathbf{M}}$ is called a **majority term**. For example, on any algebra with an underlying lattice structure, the median,

$$m(x, y, z) := (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$$

is a majority term since it satisfies the identities

$$m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx x.$$

The significance of a near-unanimity term from our perspective is that its presence gives us the full benefit of brute force at a very small price.

2.3.3 NU Lemma. (K. Baker and A. Pixley [3]) *Let $k \geqslant 2$ and assume that $\underline{\mathbf{M}}$ has a $(k+1)$-ary near-unanimity term. Let X be a subset of M^m and let $\alpha\colon X \to M$ be a map that preserves every k-ary relation in $\mathcal{B}$. Then α preserves every relation in $\mathcal{B}$.*

The theorem below is extremely useful. We shall see in Chapter 3 that it always can be extended to a full duality and in Chapter 10 that it has a very strong converse.

2.3.4 NU Duality Theorem. *Let $k \geqslant 2$ and assume that $\underline{\mathbf{M}}$ has a $(k+1)$-ary near-unanimity term. Then $\underset{\sim}{\mathbf{M}} = \langle M; \mathbb{S}(\underline{\mathbf{M}}^k), \mathcal{T}\rangle$ satisfies* (IC)*, yields a duality on $\mathcal{A}$ and is injective in $\mathfrak{X}$.*

2.3.5 Distributive Lattices Revisited. We saw in Chapter 1 that the structure $\underset{\sim}{\mathbf{D}} := \langle\{0,1\}; 0, 1, \leqslant, \mathcal{T}\rangle$ yields a full duality on the class $\mathfrak{D} = \mathbb{ISP}(\underline{\mathbf{D}})$ of distributive lattices. The fact that $\underset{\sim}{\mathbf{D}}$ yields a duality is a very simple consequence of the theory we have developed so far. Since $\underline{\mathbf{D}}$ is a lattice, it has a majority term and thus the NU Duality Theorem applies with $k = 2$. By inspection, we can see that the only binary algebraic relations on $\underline{\mathbf{D}}$ are

$$\{(0,0)\},\ \{(0,1)\},\ \{(1,0)\},\ \{(1,1)\},\ \{0\} \times D,$$
$$\{1\} \times D,\ D \times \{0\},\ D \times \{1\},\ \Delta_D,\ \leqslant,\ \geqslant,\ D^2.$$

Let $X \subseteq D^n$ and let $\alpha\colon X \to D$ be a map. Then α must preserve the trivial relations Δ_D and D^2. Moreover, it is easily seen that if α preserves the constants 0 and 1 then it will preserve the first eight relations in the list, and, of course, if α preserves $\leqslant$ then it preserves the converse relation $\geqslant$. Consequently, if α preserves 0, 1, and $\leqslant$, then it preserves every subalgebra of $\underline{\mathbf{D}}^2$. Since $\mathbb{S}(\underline{\mathbf{D}}^2)$ yields a duality on $\mathfrak{D}$, we see at once that the structure $\underset{\sim}{\mathbf{D}} = \langle D; 0, 1, \leqslant, \mathcal{T}\rangle$ yields a duality on $\mathfrak{D}$. This establishes the algebraic half of Priestley's duality.

Refining a duality via entailment. Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ and let s be a fixed finitary algebraic relation or (partial) operation on $\underline{\mathbf{M}}$. Given $\mathbf{A} \in \mathcal{A}$, we say that $G \cup H \cup R$, or simply $\underset{\sim}{\mathbf{M}}$, **entails** s **on** $D(\mathbf{A})$ if every continuous map α from $D(\mathbf{A})$ to M which preserves the relations and (partial) operations in $G \cup H \cup R$ also preserves s. The set $G \cup H \cup R$ is said to **entail** s, in symbols, $G \cup H \cup R \vdash s$, if $G \cup H \cup R$ entails s on $D(\mathbf{A})$ for all $\mathbf{A} \in \mathcal{A}$. Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ and $\underset{\sim}{\mathbf{M}}' = \langle M; G', H', R', \mathcal{T}\rangle$. If $G \cup H \cup R \vdash s$ for all $s \in G' \cup H' \cup R'$, then we say that $G \cup H \cup R$ **entails** $G' \cup H' \cup R'$, or $\underset{\sim}{\mathbf{M}}$ **entails** $\underset{\sim}{\mathbf{M}}'$, and write $(G \cup H \cup R) \vdash (G' \cup H' \cup R')$. The following lemma is a simple but frequently used consequence of the definition.

2.4.2 $\underset{\sim}{\mathbf{M}}$-Shift Duality Lemma. *If $\underset{\sim}{\mathbf{M}}$ entails $\underset{\sim}{\mathbf{M}}'$ and $\underset{\sim}{\mathbf{M}}'$ yields a duality on some subclass $\mathcal{C}$ of $\mathcal{A}$, then $\underset{\sim}{\mathbf{M}}$ also yields a duality on $\mathcal{C}$.*

This lemma allows us to combine the Brute Force Duality Theorem and the Duality Compactness Theorem 2.2.11.

2.4.3 Duality and Entailment Theorem. *Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$.*

(i) *If $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$, then $G \cup H \cup R$ entails every finitary algebraic relation and every finitary algebraic (partial) operation on $\underline{\mathbf{M}}$.*

(ii) *If $\underset{\sim}{\mathbf{M}}$ is of finite type, then the following are equivalent:*
   (a) *$\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$;*
   (b) *$G \cup H \cup R$ entails every finitary algebraic relation on $\underline{\mathbf{M}}$;*
   (c) *$G \cup H \cup R$ entails every finitary algebraic partial operation on $\underline{\mathbf{M}}$.*

In practice, when we prove that $G \cup H \cup R$ entails s it is usual to apply the following observation (which is a trivial consequence of the fact that $D(\mathbf{A})$ is a closed substructure of $\underset{\sim}{\mathbf{M}}^A$) and establish a slightly stronger but simpler condition.

2.4.4 Lemma. *Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$. Then $G \cup H \cup R$ entails s provided that, for each non-empty set S and each closed substructure $\mathbf{X}$ of $\underset{\sim}{\mathbf{M}}^S$, every morphism α from $\mathbf{X}$ to $\underset{\sim}{\mathbf{M}}$ preserves s.*

If we have some finite family Ω of relations which yields a duality on $\mathcal{A}$, typically $\Omega = \mathbb{S}(\underline{\mathbf{M}}^2)$, we would like to delete relations one at a time from Ω until we obtain a minimal subset R such that $R \vdash s$ for all $s \in \Omega \backslash R$. In this way, we obtain minimal subsets of Ω which yield a duality on $\mathcal{A}$. Such dualities will be referred to as **optimal dualities** and will be studied in detail in Chapter 8.

2.4.5 Constructs for Entailment. By an **admissible construct** of type $\langle G, H, R\rangle$ we mean a rule which associates each structure $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ with an algebraic operation, partial operation or relation s that is entailed by $\underset{\sim}{\mathbf{M}}$. A set $\mathscr{C}$ of admissible constructs will be called **complete** if s can be constructed from a finite subset of $G \cup H \cup R$, by applying a finite number of the constructs in $\mathscr{C}$, whenever $G \cup H \cup R \vdash s$. For the moment we will give an extensive list of constructs that is sufficient to establish the examples of Chapter 4. While this list of constructs is not complete in general, it can be made complete by the addition of three further constructs (see the Completeness of Entailment Theorem 9.2.6). Many of the examples $\underline{\mathbf{M}}$ considered in Chapter 4 have the property that every non-trivial subalgebra of $\underline{\mathbf{M}}$ is subdirectly irreducible and $\underline{\mathbf{M}}$ generates a congruence distributive variety—for such algebras the constructs listed below are complete (see Theorem 9.3.4).

(1) **Trivial relations** If θ is an equivalence relation on $\{1, \dots, n\}$ then any set of relations entails the relation $\Delta^\theta := \{(c_1, \dots, c_n) \in M^n \mid i \,\theta\, j \Rightarrow c_i = c_j\}$. This is an example of a nullary construct and includes as special cases the

relation M^2 and the relation $\Delta_M := \{(c,c) \mid c \in M\}$, which may be viewed as the equality relation on M or as the graph of the endomorphism $\mathrm{id}_{\underline{\mathbf{M}}}$.

(2) **Subscript manipulation** For any map $\varepsilon\colon \{1,\dots,m\} \to \{1,\dots,n\}$, the n-ary relation $r^{\varepsilon} := \{(c_1,\dots,c_n) \in M^n \mid (c_{\varepsilon(1)},\dots,c_{\varepsilon(m)}) \in r\}$ is entailed by the m-ary relation r (provided that r^{ε} is non-empty). In most applications we shall be concerned only with relations which are at most binary. Apart from the two identity maps, $\mathrm{id}_{\{1\}}$ and $\mathrm{id}_{\{1,2\}}$, there are six choices for ε when we restrict to $m,n \in \{1,2\}$: if r is unary, we can construct the binary relations $r \times M$ and $M \times r$; if r is binary we can construct the unary relation $r^1 := \pi_1(r \cap \Delta)$ and the binary relations $r^1 \times M$, $M \times r^1$, and, finally, the **converse** $r^{\smile} := \{(c_2,c_1) \mid (c_1,c_2) \in r\}$ of r.

(3) **Trivial expansion** This is the special case of subscript manipulation in which ε is one-to-one and order-preserving. For example, when $\varepsilon\colon \{1,2\} \to \{1,2,3\}$ is given by $\varepsilon(1) = 1$, $\varepsilon(2) = 3$, we obtain the ternary relation $r^{\varepsilon} = \{(c_1,c_2,c_3) \mid (c_1,c_3) \in r\}$.

(4) **Permutation** This is the special case of subscript manipulation in which $m = n$ and ε is a permutation. The converse, $r^{\smile}$, of a binary relation r is obtained when $m = n = 2$ and ε is the 2-cycle (1 2).

(5) **Repetition removal** If r is an n-ary relation and, for fixed i and j, we have $c_i = c_j$ for all $(c_1,\dots,c_n) \in r$, then $r'_j := \{(c_1,\dots,c_{j-1},c_{j+1},\dots,c_n) \in M^{n-1} \mid (c_1,\dots,c_n) \in r\}$ is entailed by r. This is also a special case of subscript manipulation.

(6) **Intersection** If r and s are n-ary relations, then $r \cap s$ is entailed by $\{r,s\}$ (provided $r \cap s$ is non-empty).

(7) **Product** Let r be n-ary and s be m-ary. For $\boldsymbol{c} = (c_1,\dots,c_n)$ and $\boldsymbol{d} = (d_1,\dots,d_m)$, let $(\boldsymbol{c},\boldsymbol{d})$ denote the corresponding $(n+m)$-tuple. Then $\{r,s\}$ entails the relation $r \times s := \{(\boldsymbol{c},\boldsymbol{d}) \in M^{n+m} \mid \boldsymbol{c} \in r \text{ and } \boldsymbol{d} \in s\}$.

(8) **Coordinate projections** Any set of relations entails the ith coordinate projection $\pi_i\colon M^n \to M$.

(9) **Restriction of domain** If $h\colon r \to M$ is an n-ary partial operation and $s \subseteq r$, then $\{h,s\}$ entails $h{\restriction_s}\colon s \to M$.

(10) **Graph** It follows immediately from Lemma 2.1.1 that a (partial) operation, h, entails its graph, $\mathrm{graph}(h)$, and conversely, if r is the graph of an n-ary partial operation, $\mathrm{graph}^{-1}(r)$, then r entails $\mathrm{graph}^{-1}(r)$. In particular, a nullary operation c entails the unary relation $\{c\}$ and conversely.

(11) **Composition** If g is a k-ary partial operation and $h_1,\dots,h_k$ are n-ary partial operations, then $\{g,h_1,\dots,h_k\}$ entails the composite $g(h_1,\dots,h_k)$. Recall that if g is a nullary operation, then the composition of g with the empty set of n-ary operations is the n-ary constant operation corresponding to g. Thus $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ entails every member of its enriched partial

clone $[G \cup H]$. Note that this is exactly the set of partial operations which are interpretations in $\underset{\sim}{\mathbf{M}}$ of the terms of type $\langle G, H, R\rangle$.

(12) **Domain** If h is a partial operation, then h entails $\mathrm{dom}(h)$.

(13) **Equaliser** If h_1 and h_2 are n-ary partial operations, then $\{h_1, h_2\}$ entails the relation $\mathrm{eq}(h_1, h_2) := \{\, \boldsymbol{c} \in \mathrm{dom}(h_1) \cap \mathrm{dom}(h_2) \mid h_1(\boldsymbol{c}) = h_2(\boldsymbol{c})\}$ (provided the latter is non-empty). A useful special case is the **fixpointset**, $\mathrm{fix}(e) := \{\, c \in N \mid e(c) = c\,\} = \mathrm{eq}(e, \mathrm{id}_{\underline{\mathbf{M}}})$, of a **partial endomorphism** $e\colon \mathbf{N} \to \underline{\mathbf{M}}$, where $\mathbf{N}$ is a subalgebra of $\underline{\mathbf{M}}$.

(14) **Joint kernel** If h_1 and h_2 are respectively n-ary and m-ary partial operations, then the set $\{h_1, h_2\}$ entails the $(n+m)$-ary relation $\ker(h_1, h_2) := \{(\boldsymbol{c}, \boldsymbol{d}) \in \mathrm{dom}(h_1) \times \mathrm{dom}(h_2) \mid h_1(\boldsymbol{c}) = h_2(\boldsymbol{d})\}$ (provided this set is non-empty).

(15) **Action by a partial endomorphism** If r is an n-ary relation and e is a partial endomorphism of $\underline{\mathbf{M}}$, then the set $\{e, r\}$ entails
$$e \cdot r := \{(c_1, \dots, c_n) \in M^n \mid c_1 \in \mathrm{dom}(e) \;\&\; (e(c_1), c_2, \dots, c_n) \in r\}$$
(provided this set is non-empty). If r is a unary relation, then this is simply $e^{-1}(r) := \{\, c \in \mathrm{dom}(e) \mid e(c) \in r\,\}$. We then have $\{e, r\} \vdash e^{-1}(r)$.

In Exercise 2.6 we ask the reader to check the soundness of these constructs.

2.4.6 Soundness Theorem. *Each of the constructs from* (1) *to* (15) *is admissible.*

One construct which is notable by its absence is the **relational product** of an n-ary relation r and an m-ary relation s:
$$r \cdot s := \{(c_1, \dots, c_{n+m-2}) \in M^{n+m-2} \mid$$
$$(\exists c \in M)(c_1, \dots, c_{n-1}, c) \in r \text{ and } (c, c_n, \dots, c_{n+m-2}) \in s\,\}.$$
In general, relational product is not an admissible construct (see 9.1.3).

Update 2

Several of the main results of this chapter extend to the case where $\underline{\mathbf{M}}$ is allowed to be an arbitrary finite structure. In particular, the Duality Compactness Theorem 2.2.11 is extended using the theory of sketches in [47], while the First Duality Theorem 2.2.2, Lemma 2.2.5, and the Second Duality Theorem 2.2.7 are extended in [17]. Some results extend only under the assumption that the type of $\underline{\mathbf{M}}$ includes no partial operations; for example, the important NU Duality Theorem 2.3.4 [17].

CHAPTER 3: STRONG DUALITIES

We say that $\underset{\sim}{\mathbf{M}}$ yields a **full duality** on $\mathcal{A}$ if it yields a duality on $\mathcal{A}$ and, for each $\mathbf{X} \in \mathcal{X} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$,

$$\varepsilon_{\mathbf{X}} \colon \mathbf{X} \to DE(\mathbf{X})$$

is surjective and therefore is an isomorphism. Thus $\underset{\sim}{\mathbf{M}}$ yields a full duality on $\mathcal{A}$ exactly when it yields a duality on $\mathcal{A}$ which is a dual equivalence.

Our goal in this chapter will be to develop a number of methods for modifying a choice of $\underset{\sim}{\mathbf{M}}$ which is already known to yield a duality on $\mathcal{A}$, to obtain one which yields a full duality on $\mathcal{A}$. This will always be done by adding operations as well as partial operations to its structure. In particular, we will see just why *partial* operations are in general necessary to do this.

Full duality and the dual category. In the context of full duality a special role is played by the (possibly empty) set K of elements of M which determine one-element subalgebras of $\underline{\mathbf{M}}$. If $\underset{\sim}{\mathbf{M}}$ is algebraic over $\underline{\mathbf{M}}$, then the Preduality Theorem 1.5.2(iv) assures us that K will determine a substructure $\mathbf{K}$ of $\underset{\sim}{\mathbf{M}}$. Under a full duality we can say much more about $\mathbf{K}$. Let $\underline{\mathbf{1}}$ denote the one-element algebra $\underline{\mathbf{M}}^{\varnothing}$ in $\mathcal{A}$ and recall that $[G \cup H]$ denotes the enriched partial clone generated by $G \cup H$, that is, the enriched partial clone of $\underset{\sim}{\mathbf{M}}$.

3.1.2 Lemma. *Assume that $\underset{\sim}{\mathbf{M}}$ yields a full duality on $\mathcal{A}$.*

(i) $\mathbf{K}$ *and* $\underline{\mathbf{1}}$ *are dual to one another:* $E(\mathbf{K}) \cong \underline{\mathbf{1}}$ *and* $D(\underline{\mathbf{1}}) \cong \mathbf{K}$.

(ii) $\mathbf{K}$ *is the substructure of* $\underset{\sim}{\mathbf{M}}$ *generated by its distinguished elements. Indeed, the following are equivalent:*

  (a) c *determines a one-element subalgebra of* $\underline{\mathbf{M}}$*, that is,* $c \in K$*;*

  (b) c *is the value of a constant unary function in* $[G \cup H]$*;*

  (c) c *is the value of a constant unary total function in* $[G \cup H]$*;*

  (d) c *is the value of a nullary term in* $[G \cup H]$*;*

  (e) c *is in the substructure of* $\underset{\sim}{\mathbf{M}}$ *generated by its distinguished elements.*

(iii) *For every* $\mathbf{X} \in \mathcal{X}$ *there is a unique embedding of* $\mathbf{K}$ *into* $\mathbf{X}$.

(iv) $\mathbf{K}$ *is an initial object (free object on the empty set) in* $\mathcal{X}$ *while* $\underline{\mathbf{1}}$ *is a final object in* $\mathcal{A}$.

(v) $\underset{\sim}{\varnothing} \in \mathcal{X}$ *if and only if* $\mathbf{K} = \underset{\sim}{\varnothing}$ *if and only if* $\underset{\sim}{\mathbf{M}}$ *has no nullary operations if and only if* $\underline{\mathbf{1}} \notin \mathbb{IS}\,\underline{\mathbf{M}}$.

In view of (ii) above, in order to achieve full duality, we are obliged to include as distinguished elements of $\underset{\sim}{\mathbf{M}}$ enough members of $\mathbf{K}$ to generate all of $\mathbf{K}$ in $\underset{\sim}{\mathbf{M}}$. For $\mathbf{X} \in \mathcal{X}$ we will denote by $\mathbf{K}_{\mathbf{X}}$ the image of $\mathbf{K}$ under the unique embedding of (iii) above.

Assume that we begin with a choice of $\underset{\sim}{\mathbf{M}}$ that yields a duality on $\mathcal{A}$. Then D and E give us a dual equivalence between $\mathcal{A}$ and the subcategory $\mathbb{I}(D(\mathcal{A}))$ of $\mathcal{X} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$. Now suppose that we augment $G \cup H$ with additional algebraic (partial) operations. The new structure $\underset{\sim}{\mathbf{M}}'$ will then entail $\underset{\sim}{\mathbf{M}}$, and therefore will also yield a duality on $\boldsymbol{\mathcal{A}}$. Again D and E give us a dual equivalence between $\boldsymbol{\mathcal{A}}$ and the subcategory $\mathbb{I}(D(\boldsymbol{\mathcal{A}}))$ of $\mathcal{X}' = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})'$. But now $\mathcal{X}'$ will be a smaller extension of $\mathbb{I}(D(\boldsymbol{\mathcal{A}}))$ since we have eliminated the substructures of powers of $\underset{\sim}{\mathbf{M}}$ that are not closed under the new (partial) operations. By adding enough operational structure to $\underset{\sim}{\mathbf{M}}$ we would like to squeeze the category $\mathcal{X}$ down until it coincides with the category $\mathbb{I}(D(\boldsymbol{\mathcal{A}}))$, thus giving us a full duality.

It turns out that this strategy can be further extended, as the members of $D(\boldsymbol{\mathcal{A}})$ are closed under more than just the *finitary* algebraic (partial) operations. Let I be an arbitrary set, $\mathbf{B}$ a subalgebra of $\underline{\mathbf{M}}^I$ and $h\colon \mathbf{B} \to \underline{\mathbf{M}}$ a homomorphism, that is, h is an **algebraic I-ary partial operation** on $\underline{\mathbf{M}}$. Just as we do in the finitary case, we may extend the map h pointwise to an I-ary partial operation $\boldsymbol{h}$ on any power M^S of M. For each $s \in S$, let $\pi_s\colon M^S \to M$ denote the sth **projection** given by $\pi_s(y) = y(s)$ for each $y \in M^S$. Then the domain of the extension $\boldsymbol{h}$ is

$$\operatorname{dom}(\boldsymbol{h}) = \{\, \boldsymbol{x} \in (M^S)^I \mid \pi_s \circ \boldsymbol{x} \in B \text{ for all } s \in S \,\} \subseteq (M^S)^I$$

and $\boldsymbol{h}\colon \operatorname{dom}(\boldsymbol{h}) \to \underline{\mathbf{M}}^S$ is defined by $(\boldsymbol{h}(\boldsymbol{x}))(s) = h(\pi_s \circ \boldsymbol{x})$ for $\boldsymbol{x} \in \operatorname{dom}(\boldsymbol{h})$. As is customary, we say that a subset X of M^S is **closed under** h provided $\boldsymbol{h}(\boldsymbol{x}) \in X$ whenever $\boldsymbol{x} \in \operatorname{dom}(\boldsymbol{h})$ and $\boldsymbol{x}(i) \in X$ for each $i \in I$. We shall say that X is **hom-closed (in M^S)** if, for each set I, the set X is closed under every algebraic I-ary partial operation h on $\underline{\mathbf{M}}$. In particular, taking $I = \varnothing$, if X is hom-closed, then it contains the constant map $\boldsymbol{a}\colon S \to M$ onto $\{a\}$ whenever $\{a\}$ forms a one-element subalgebra of $\underline{\mathbf{M}}$. Note that $\varnothing$ is hom-closed in $\underline{\mathbf{M}}^S$ if and only if $\underset{\sim}{\varnothing}$ is in $\mathcal{X}$.

We will see in Corollary 3.1.5 that the members of $D(\boldsymbol{\mathcal{A}})$ can be characterised up to isomorphism as the hom-closed subsets of powers of M. This will be more easily accomplished by first giving an alternative and equally important description of $\mathbb{I}D(\boldsymbol{\mathcal{A}})$ in terms of the $\boldsymbol{\mathcal{A}}$-free algebras. For an arbitrary non-empty set S, the subalgebra of $\underline{\mathbf{M}}^{M^S}$ ($\boldsymbol{\mathcal{A}}$-freely) generated by the projections $\{\, \pi_s \mid s \in S \,\}$ is called the **free algebra of S-ary term functions** over $\underline{\mathbf{M}}$ and is denoted by $\mathbf{F}_{\underline{\mathbf{M}}}(S)$. We say that $y \in M^S$ is in the **term closure** of a set $X \subseteq M^S$ if any two S-ary term functions that agree on X also agree at y. (See Appendix A for a review of free term algebras and closure operations.) Thus a subset X of M^S is **term-closed (in M^S)** if for all $y \in M^S \backslash X$ there exist S-ary term functions $\sigma, \tau\colon M^S \to M$ on $\underline{\mathbf{M}}$ that agree on X but not at y. Alternatively, X is term-closed in M^S provided it is an intersection of equaliser sets

$$\operatorname{eq}(\sigma, \tau) = \{\, x \in M^S \mid \sigma(x) = \tau(x)\}$$

of pairs (σ, τ) of S-ary term functions from $\mathbf{F}_{\underline{\mathbf{M}}}(S)$. Note that $\varnothing$ is term-closed in M^S if and only if the conditions of Lemma 3.1.2(v) hold.

Despite their disparate origins, these two notions of closure prove to be equivalent.

3.1.3 Closure Theorem. *If $S \neq \varnothing$ and $X \subseteq M^S$, then X is term-closed in M^S if and only if X is hom-closed in M^S.*

The fact that condition (i) of the Preduality Theorem 1.5.2 extends to hom-closure is more easily verified by looking at term closure.

3.1.4 Lemma. *$D(\mathbf{A}) = \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ is term-closed ($=$ hom-closed) in M^A for all $\mathbf{A} \in \mathcal{A}$.*

3.1.5 Corollary. *Let $\mathbf{X}$ be in $\mathcal{X}$. Then $\mathbf{X} \in \mathbb{I}(D(\mathcal{A}))$ if and only if $\mathbf{X}$ is isomorphic to a term-closed ($=$ hom-closed) subset of $\underset{\sim}{\mathbf{M}}^S$ for some non-empty set S.*

We can now prove a variant of the First Duality Theorem 2.2.2 which is more exactly suited to our needs. Here we have replaced (INJ) with a different collection (INJ)$'$ of instances of injectivity of $\underset{\sim}{\mathbf{M}}$ in $\mathcal{X}$ expressed fully in terms of the category $\mathcal{X}$.

3.1.6 Third Duality Theorem. *The following are equivalent:*

(i) *$\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$;*

(ii) *if $\mathbf{X}$ is a term-closed substructure of $\underset{\sim}{\mathbf{M}}^S$ for some $S \neq \varnothing$, then $E(\mathbf{X}) = \mathbf{F}_{\underline{\mathbf{M}}}(S){\restriction}_X$, that is, every morphism $\alpha \colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ extends to an S-ary term function $\tau \colon \underset{\sim}{\mathbf{M}}^S \to \underset{\sim}{\mathbf{M}}$;*

(iii) *the following two conditions hold—*

(INJ)$'$ *$\underset{\sim}{\mathbf{M}}$ is injective with respect to term-closed sets $\mathbf{X} \subseteq \underset{\sim}{\mathbf{M}}^S$ (with $S \neq \varnothing$) and their inclusion maps, that is, each morphism $\alpha \colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ extends to a morphism $\beta \colon \underset{\sim}{\mathbf{M}}^S \to \underset{\sim}{\mathbf{M}}$,*

((CLO) *for each $n \in \mathbb{N}$, every morphism $\tau \colon \underset{\sim}{\mathbf{M}}^n \to \underset{\sim}{\mathbf{M}}$ is an n-ary term function on $\underline{\mathbf{M}}$.*

Moreover, the same is true if $\mathcal{A}$ is replaced by $\mathcal{A}_{\mathit{fin}}$ in (i) *and S is restricted to finite sets in* (ii) *and* (iii).

In practice we will normally begin with a duality which we seek to upgrade to a full duality. Our characterisation of the dual category tells us just what will be required to do this.

3.1.7 Full Duality Theorem. *If $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$, then the following are equivalent:*

(i) *$\underset{\sim}{\mathbf{M}}$ yields a full duality on $\mathcal{A}$;*

(ii) *$\mathcal{X} = \mathbb{I}D(\mathcal{A})$;*

(iii) *every closed substructure of a power of $\underset{\sim}{\mathbf{M}}$ is isomorphic to a term-closed ($=$ hom-closed) substructure of a power of $\underset{\sim}{\mathbf{M}}$.*

Moreover, the same is true if $\mathcal{A}$ and $\mathcal{X}$ are replaced by $\mathcal{A}_{fin}$ and $\mathcal{X}_{fin}$ in (i) *and* (ii)*, and* (iii) *is restricted to finite powers of $\utilde{\mathbf{M}}$.*

Strong duality and the role of injectivity. We can at last reveal the exact goal of this chapter. Assume that $\utilde{\mathbf{M}}$ yields a duality on $\mathcal{A}$. To establish that $\utilde{\mathbf{M}}$ yields a full duality on $\mathcal{A}$, the Full Duality Theorem 3.1.7 says that we must find, for each non-empty set S and each closed substructure $\mathbf{X}$ of $\utilde{\mathbf{M}}^S$, a non-empty set T and term-closed (=hom-closed) substructure $\mathbf{Y}$ of $\utilde{\mathbf{M}}^T$ such that $\mathbf{X}$ is isomorphic to $\mathbf{Y}$. But where, given $\mathbf{X}$, are we to find the structure $\mathbf{Y}$? This seemingly daunting task is actually carried out in every known example of full duality by establishing that the natural candidate $\mathbf{Y} = \mathbf{X}$ will suffice, that is, that every closed substructure of a power of $\utilde{\mathbf{M}}$ *is* term-closed (or, equivalently, that it *is* hom-closed). When $\utilde{\mathbf{M}}$ yields a duality on $\mathcal{A}$ and every closed substructure of a power of $\utilde{\mathbf{M}}$ is term-closed, we say that $\utilde{\mathbf{M}}$ **yields a strong duality on** $\mathcal{A}$.

As we have said, every known full duality can be shown to be full by showing that it is in fact strong. In every known full duality, (INJ) has been established by proving that $\utilde{\mathbf{M}}$ is in fact injective in $\mathcal{X}$. It turns out that these two enhancements of full duality are equivalent. A more category-theoretic proof of the following result is presented in Exercises 3.13 and 3.14.

3.2.4 First Strong Duality Theorem. *$\utilde{\mathbf{M}}$ yields a strong duality on $\mathcal{A}$ if and only if $\utilde{\mathbf{M}}$ yields a full duality on $\mathcal{A}$ and is injective in $\mathcal{X}$. The corresponding result holds at the finite level, that is, $\utilde{\mathbf{M}}$ yields a strong duality on $\mathcal{A}_{fin}$ if and only if $\utilde{\mathbf{M}}$ yields a full duality on $\mathcal{A}_{fin}$ and is injective in $\mathcal{X}_{fin}$.*

The First Duality Theorem 2.2.2 and the Third Duality Theorem 3.1.6 say that many instances of the injectivity of $\utilde{\mathbf{M}}$ in $\mathcal{X}$ must hold once we have a duality. If duality is to be established by means of the Second Duality Theorem 2.2.7, then $\utilde{\mathbf{M}}$ must be injective in $\mathcal{X}$. Nevertheless, it is not known whether or not the injectivity of $\utilde{\mathbf{M}}$ in $\mathcal{X}$ is actually required for $\utilde{\mathbf{M}}$ to yield a full duality, and this remains one of the oldest and most tantalising open questions in the foundations of duality theory.

3.2.7 Full vs Strong Problem. [4] *Does there exist a finite algebra $\underline{\mathbf{M}}$ and a choice of $\utilde{\mathbf{M}}$ such that $\utilde{\mathbf{M}}$ yields a duality on $\mathcal{A} = \mathbb{ISP}(\underline{\mathbf{M}})$ which is full but not strong, or equivalently, such that $\utilde{\mathbf{M}}$ yields a full duality on $\mathcal{A}$ with $\utilde{\mathbf{M}}$ non-injective in $\mathcal{X}$?*

In every known full duality, $\utilde{\mathbf{M}}$ is injective in $\mathcal{X}$. The corresponding statement about $\underline{\mathbf{M}}$ is false: the four-element Heyting algebra chain has an internal isomorphism which does not extend to an endomorphism (Theorem 4.2.3(iii)).

[4]This problem was answered in the affirmative in [13]. A characterisation of finite algebras for which full implies strong remains elusive, though much is known; see [25, 19, 20, 33] , for example.

Recall that $\underset{\sim}{\mathbf{M}}$ is a **total structure** if $H = \varnothing$, that is, if every operation of $\underset{\sim}{\mathbf{M}}$ is total rather than partial.

3.2.9 Second Strong Duality Theorem. *If $\underset{\sim}{\mathbf{M}}$ is a total structure which yields a duality on $\mathcal{A}$, then the following are equivalent:*

(i) $\underset{\sim}{\mathbf{M}}$ *yields a strong duality on $\mathcal{A}$;*
(ii) $\underset{\sim}{\mathbf{M}}$ *yields a strong duality on $\mathcal{A}_{\mathrm{fin}}$;*
(iii) $\underset{\sim}{\mathbf{M}}$ *satisfies the* ***Finite Term Closure*** *condition—*
(FTC) *If $\mathbf{X}$ is a substructure of $\underset{\sim}{\mathbf{M}}^n$ for some $n \in \mathbb{N}$ and $y \in M^n \backslash X$, then there exist term functions $\sigma, \tau \colon \underset{\sim}{\mathbf{M}}^n \to \underset{\sim}{\mathbf{M}}$ on $\underline{\mathbf{M}}$ (that is, morphisms) which agree on X but not at y.*

While Lemmas 3.2.6 and 3.2.8 highlight the symmetry between $\underline{\mathbf{M}}$ in $\mathcal{A}$ and $\underset{\sim}{\mathbf{M}}$ in $\mathcal{X}$ which has characterised our development of duality theory, this symmetry begins to break down in the presence of partial operations in $\underset{\sim}{\mathbf{M}}$. For example, partial operations preclude a full converse to our next lemma.

3.2.10 Injectivity Lemma. *Assume that $\underset{\sim}{\mathbf{M}}$ yields a full duality on $\mathcal{A}$. If $\underline{\mathbf{M}}$ is injective in $\mathcal{A}$, then $\underset{\sim}{\mathbf{M}}$ is injective in $\mathcal{X}$. The converse is true provided that for every morphism $\varphi \colon \mathbf{X} \to \mathbf{Y}$ in $\mathcal{X}$, the image of $\mathbf{X}$ under φ is a substructure of $\mathbf{Y}$. In particular this is the case when $\underset{\sim}{\mathbf{M}}$ is a total structure.*

We can now give purely finite conditions for the existence of a strong duality when $\underset{\sim}{\mathbf{M}}$ is a total structure and R is finite by combining the Second Duality Theorem 2.2.7 and the Second Strong Duality Theorem 3.2.9.

3.2.11 Third Strong Duality Theorem. *Assume that the structure $\underset{\sim}{\mathbf{M}} = \langle M; G, R, \mathcal{T}\rangle$ is total and that R is finite. Then the following are equivalent:*

(i) $\underset{\sim}{\mathbf{M}}$ *yields a strong duality on $\mathcal{A}$;*
(ii) $\underset{\sim}{\mathbf{M}}$ *yields a strong duality on $\mathcal{A}_{\mathrm{fin}}$;*
(iii) (IC) *and* (FTC) *hold.*

Producing strong dualities. In this section we will gather together the theory that we have developed so far and use it to prove two theorems: the Two-for-One Strong Duality Theorem and the NU Strong Duality Theorem, together with several useful corollaries of each. Until very recently these two results could be used to produce all known full dualities, all of which are in fact strong dualities. The Two-for-One Strong Duality Theorem is derived from our description of the dual category $\mathbb{I}D(\mathcal{A})$ as the isomorphs of term-closed sets, while the NU Strong Duality Theorem derives from our characterisation of $\mathbb{I}D(\mathcal{A})$ as the isomorphs of hom-closed sets. It is here that each of these two divergent approaches comes to fruition.

If $\underset{\sim}{\mathbf{M}} = \langle M; G, \mathcal{T}\rangle$, that is, the sets H of partial operations and R of relations are empty, then we say that $\underset{\sim}{\mathbf{M}}$ is a **total algebra**. We say that an algebra $\mathbf{A}$

has named constants if the value of every constant unary term function on $\mathbf{A}$ is the value of a nullary term. The following is an immediate consequence of Lemma 3.1.2.

3.3.1 Lemma. *If a total algebra $\underset{\sim}{\mathbf{M}} = \langle M; G, \mathcal{T}\rangle$ yields a full duality on $\mathcal{A}$, then the algebra $\underline{\mathbf{M}}' = \langle M; G\rangle$ has named constants.*

3.3.2 Two-for-One Strong Duality Theorem. *Assume that both $\underline{\mathbf{M}} = \langle M; F\rangle$ and $\underline{\mathbf{M}}' = \langle M; G\rangle$ have named constants. Consider the total algebras $\underset{\sim}{\mathbf{M}} = \langle M; G, \mathcal{T}\rangle$ and $\underset{\sim}{\mathbf{M}}' = \langle M; F, \mathcal{T}\rangle$ and define $\mathcal{A}' = \mathbb{ISP}(\underline{\mathbf{M}})'$ and $\mathfrak{X}' = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}}')$. Then the following are equivalent:*

- (i) $\underset{\sim}{\mathbf{M}}$ *yields a strong duality on* $\mathcal{A}$*;*
- (ii) $\underset{\sim}{\mathbf{M}}$ *and* $\underset{\sim}{\mathbf{M}}'$ *yield strong dualities on* $\mathcal{A}$ *and* $\mathcal{A}'$ *respectively;*
- (iii) (IC) *and* (FTC) *hold with respect to* $\underset{\sim}{\mathbf{M}}$*;*
- (iv) (IC) *and* (FTC) *hold with respect to* $\underset{\sim}{\mathbf{M}}'$*;*
- (v) (FTC) *holds with respect to both* $\underset{\sim}{\mathbf{M}}$ *and* $\underset{\sim}{\mathbf{M}}'$*;*
- (vi) (IC) *holds with respect to both* $\underset{\sim}{\mathbf{M}}$ *and* $\underset{\sim}{\mathbf{M}}'$*;*
- (vii) *the algebras* $\underline{\mathbf{M}}$ *and* $\underline{\mathbf{M}}'$ *satisfy the following symmetric conditions–*
  - (a) *every homomorphism from a subalgebra of* $\underline{\mathbf{M}}^n$ *into* $\underline{\mathbf{M}}$ *extends to an* n*-ary term function on* $\underline{\mathbf{M}}'$*, and*
  - (b) *every homomorphism from a subalgebra of* $(\underline{\mathbf{M}}')^n$ *into* $\underline{\mathbf{M}}'$ *extends to an* n*-ary term function on* $\underline{\mathbf{M}}$*.*

Underlying our quest for strong dualities is a fundamental question: can every duality be upgraded to a strong duality? We state this question more precisely below.

3.3.4 Strong Upgrade Problem. [5] *Assume that $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$. Can we extend the structure $\underset{\sim}{\mathbf{M}}$ by adding* (*finitely many*) *partial or total operations and so obtain a structure $\underset{\sim}{\mathbf{M}}'$ which yields a strong duality* (*or a full duality*) *on $\mathcal{A}$?*

Consider, for example, a finite algebra $\underline{\mathbf{M}}$ which is a lattice under some pair of binary term functions. The NU Duality Theorem 2.3.4 gives us a duality on $\mathcal{A} = \mathbb{ISP}(\underline{\mathbf{M}})$ by taking for R (a set that entails all of) the binary algebraic relations on $\underline{\mathbf{M}}$. Can finite G and H be chosen to make this into a full duality? A strong duality? We will show that *every* duality which arises from the NU Duality Theorem 2.3.4 can be upgraded to a strong duality by adding finitely many (partial) operations to $\underset{\sim}{\mathbf{M}}$. The key is to observe that, in a congruence distributive variety, Bjarni Jónsson has already done all but a finite amount of the work for us, and that a handful of partial operations will do the rest!

[5]This problem was solved in the negative in [49], where a finite dualisable but not fully dualisable algebra is presented. An example of a finite algebra that is fully dualisable but not strongly dualisable is given in [63].

Every congruence on a finite algebra $\mathbf{Q}$ is a meet of meet-irreducible congruences on $\mathbf{Q}$ (Lemma A.9). Let $\operatorname{irr}(\mathbf{Q})$ be the least n such that the zero congruence $\mathbf{0}^{\mathbf{Q}}$ on $\mathbf{Q}$ is a meet of n meet-irreducible congruences. Then $\mathbf{Q}$ is subdirectly irreducible if and only if $\operatorname{irr}(\mathbf{Q}) = 1$. We define the **irreducibility index** of a finite algebra $\underline{\mathbf{M}}$ by

$$\operatorname{Irr}(\underline{\mathbf{M}}) = \max\{\operatorname{irr}(\mathbf{Q}) \mid \mathbf{Q} \text{ is a subalgebra of } \underline{\mathbf{M}}\}.$$

In order to state our central strong duality theorem we define $\mathcal{P}_n$ and $\mathcal{B}_n$, for $n = 1, 2, 3, \dots$, to be the sets of all n-ary (partial) operations and relations, respectively, which are algebraic over $\underline{\mathbf{M}}$. By analogy with $\mathcal{B}$, we defined $\mathcal{P} := \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$. Recall that K denotes the set of one-element subalgebras of $\underline{\mathbf{M}}$.

3.3.8 NU Strong Duality Theorem. *Let $k \geqslant 2$ and assume that $\underline{\mathbf{M}}$ has a $(k+1)$-ary near-unanimity term. Define $H = \bigcup\{\mathcal{P}_n \mid 1 \leqslant n \leqslant \operatorname{Irr}(\underline{\mathbf{M}})\}$. Then the structure $\underset{\sim}{\mathbf{M}} = \langle M; K, H, \mathcal{B}_k, \mathcal{T}\rangle$ yields a strong duality on $\boldsymbol{\mathcal{A}}$.*

Most known strong dualities are obtained as applications of this theorem. It is a powerful result, as it gives us an exact recipe for constructing from a near-unanimity term for $\underline{\mathbf{M}}$ a structure $\underset{\sim}{\mathbf{M}}$ that will yield a strong duality on $\boldsymbol{\mathcal{A}}$. Recall that a ternary near-unanimity term is called a **majority term**.

3.3.9 NU Strong Duality Corollary. *Assume that $\underline{\mathbf{M}}$ has a majority term and that all of the non-trivial subalgebras of $\underline{\mathbf{M}}$ are subdirectly irreducible. Then $\underset{\sim}{\mathbf{M}} = \langle M; K, \mathcal{P}_1, \mathcal{B}_2, \mathcal{T}\rangle$ yields a strong duality on $\boldsymbol{\mathcal{A}}$.*

The NU Strong Duality Theorem remains a highly untapped resource, as almost all interesting applications of it to date follow from the NU Strong Duality Corollary where we have the strongest special case: $k = 2$ and $\operatorname{Irr}(\underline{\mathbf{M}}) = 1$. Even the NU Strong Duality Corollary can lead to complex dual categories, for $\underline{\mathbf{M}}^2$ may have many subalgebras. In Chapter 4 we will see a range of examples in which the methods of Chapter 2 can be used to reduce these binary relations to a small and convenient number. We return to this problem in Chapters 8 and 9 where a full theory for achieving such reductions will be developed.

Update 3

A characterisation of when a duality can be upgraded to a strong duality is given in [14]—see Exercise 9.8. A useful sufficient condition for a strong upgrade, namely **having enough algebraic operations**, is given in [59]. Having enough algebraic operations is a special case of having **finite rank**, a sufficient condition introduced in [59]. In the appendix to [66], the concept of rank is sharpened to the concept of **height** and is used to give a necessary and sufficient condition for a strong upgrade.

Extensions of the results of this chapter to the case where $\underline{\mathbf{M}}$ is a finite structure are given in [47, 17]. In [47] it is proved that if the type of $\underset{\sim}{\mathbf{M}}$ contains no partial operations, then a full duality at the finite level lifts to a full duality between $\mathcal{A}$ and $\mathfrak{X}$. This is expanded on in [17] and a corresponding lifting theorem is proved for strong duality. The Two-for-One Strong Duality Theorem has been extended to the setting in which $\underline{\mathbf{M}}$ is allowed to be a finite structure. Its extends as stated to the case in which both $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}$ are total structures [47, Thm. 2.5]. Various generalisations have been obtained that allow one of $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}$ to include partial operation in its type [17, Sec. 6], [22, Thm. 2.4]—these results now come under the general heading of *topology swapping*.

When $\underline{\mathbf{M}}$ is a structure, a complication arises in connection with the notion of strong duality: while the equivalence of hom-closed and term-closed still holds, the equivalence with the injectivity of $\underset{\sim}{\mathbf{M}}$ in $\mathfrak{X}$, given in the First Strong Duality Theorem 3.2.4, fails. See [33] for a detailed discussion of full and strong duality in this setting.

CHAPTER 4: EXAMPLES OF STRONG DUALITIES

NU strong dualities. In this section we will exhibit a few of the many non-arithmetical applications of the NU Strong Duality Corollary 3.3.9. In all of these examples we have the **median operation**

$$m(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$$

generated by operations $\vee$ and $\wedge$ which are either lattice operations or close enough to lattice operations to ensure that m is a majority term.

4.3.1 Distributive Lattices. The variety $\mathfrak{D}$ of **distributive lattices** is normally defined as the models $\mathbf{L} = \langle L; \vee, \wedge \rangle$ of the equations (DL) of Chapter 1, Section 2. In Lemma 1.2.2 we found that $\mathfrak{D} = \mathbb{ISP}(\underline{\mathbf{D}})$ where

$$\underline{\mathbf{D}} = \langle \{0, 1\}; \vee, \wedge \rangle.$$

Applying the NU Strong Duality Corollary we obtain a strong duality by taking $G = \{0, 1\}$, $H = \varnothing$ and $R = \mathbb{S}(\underline{\mathbf{D}} \times \underline{\mathbf{D}})$. In the example of 2.2.5 we saw that

$$\underset{\sim}{\mathbf{D}} = \langle \{0, 1\}; 0, 1, \leqslant, \mathcal{T} \rangle$$

entails all binary algebraic relations on $\underline{\mathbf{D}}$. Using the $\underset{\sim}{\mathbf{M}}$-Shift Strong Duality Lemma 3.2.3(iii)(a) and recalling Lemma 1.2.8, we obtain a strengthened version of the Priestley Dual Equivalence Theorem 1.2.5.

4.3.2 Theorem. (Priestley [69, 70]) *$\underset{\sim}{\mathbf{D}}$ yields a strong duality between the variety $\mathcal{D}$ of distributive lattices and the category $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{D}})$ of bounded Priestley spaces.*

4.3.9 Kleene Algebras. For a different approach to three-valued logic, an algebra $\underline{\mathbf{K}} = \langle K; \vee, \wedge, \neg, 0, 1\rangle$ is called a **Kleene algebra** if it is a bounded distributive lattice satisfying the axioms

$$\neg\neg x \approx x,\ \neg 0 \approx 1,\ \neg(x \wedge y) \approx \neg x \vee \neg y,\ x \wedge \neg x \leqslant y \vee \neg y.$$

The models of these axioms form a variety $\mathcal{K} = \mathbb{ISP}(\underline{\mathbf{K}})$ generated by the three-element chain

$$\underline{\mathbf{K}} = \langle \{0, a, 1\}; \vee, \wedge, \neg, 0, 1\rangle$$

where

$$0 < a < 1,\ \neg 0 = 1,\ \neg 1 = 0 \ \text{ and } \ \neg a = a$$

(see Balbes and Dwinger [4]). In addition to the Boolean truth values of 0 (*false*) and 1 (*true*), we include the value a for *don't know*. The fact that $\neg a = a$ reflects the observation that, if we don't know if a statement is true, then we also don't know if its negation is true.

Because $\underline{\mathbf{K}}$ and its only subalgebra $\mathbf{K}_0 = \langle \{0, 1\}; \vee, \wedge, \neg, 0, 1\rangle$ are both simple and have no non-identity endomorphisms, we obtain a strong duality by taking $G = H = \varnothing$ and $R = \mathbb{S}(\underline{\mathbf{K}} \times \underline{\mathbf{K}})$. Among the relations of R we single out the order $\preccurlyeq$, illustrated in Figure 4.3, together with the unary relation K_0 and the reflexive, symmetric relation

$$\sim\, = \{(0,0), (a,a), (1,1), (0,a), (1,a), (a,0), (a,1)\}$$

relating all pairs except 0 and 1. Let

$$\underset{\sim}{\mathbf{K}} = \langle \{0, a, 1\}; \preccurlyeq, \sim, K_0, \mathcal{T}\rangle.$$

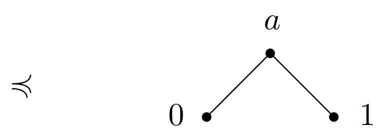

FIGURE 4.3 the order $\preccurlyeq$ on $\underset{\sim}{\mathbf{K}}$

4.3.10 Theorem. (Davey and Werner [40])

(i) *$\underset{\sim}{\mathbf{K}}$ yields a strong duality on the variety $\mathcal{K}$ of Kleene algebras.*

(ii) *$\mathbf{X} = \langle X; \preccurlyeq, \sim, X_0, \mathcal{T}\rangle$ belongs to the dual category $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{K}})$ if and only if $\langle X; \preccurlyeq\rangle$ is a Priestley space, $\sim$ is a closed binary relation, X_0 is a closed subspace and the following universal axioms are satisfied:*

   (a) $x \sim x$,

   (b) $x \sim y$ *and* $x \in X_0 \implies x \preccurlyeq y$,

   (c) $x \sim y$ *and* $y \preccurlyeq z \implies z \sim x$.

4.3.12 (CLO) versus (IC). From the example of Kleene algebras we obtain a simple illustration of the fact that a choice of relations for $\underset{\sim}{\mathbf{M}}$ which determine the clone of $\underline{\mathbf{M}}$ may not be enough to give us the duality condition (IC). Consider the structure

$$\underset{\sim}{\mathbf{K}}' = \langle \{0, a, 1\}; \preccurlyeq, K_0, \mathcal{T} \rangle.$$

If φ is a *total* operation on K which preserves $\preccurlyeq$ and K_0, it will also preserve $\succcurlyeq$ as well as the relational product $\sim = \succcurlyeq \cdot \preccurlyeq$. Since $\underset{\sim}{\mathbf{K}}$ satisfies (CLO), φ must be a term function. Thus $\underset{\sim}{\mathbf{K}}'$ also satisfies (CLO). But $\underset{\sim}{\mathbf{K}}'$ does not satisfy (IC), as we see by taking $X = \{(0, a), (a, 0)\}$ and $\gamma \colon (0, a) \mapsto 0;\ \ (a, 0) \mapsto 1$. Then $\gamma \colon \mathbf{X} \to \underset{\sim}{\mathbf{K}}'$ preserves $\preccurlyeq$ and K_0 (vacuously), but it does not preserve $\sim$ and therefore does not extend to a term function. In particular, $\underset{\sim}{\mathbf{K}}'$ determines the clone of $\underline{\mathbf{K}}$ but does not yield a duality on the quasi-variety it generates.

Kleene algebras have played an important role in the development of natural duality theory. They occur as seminal examples several times later in this text: see Section 5 of Chapter 7 and Section 4 of Chapter 8.

Update 4

The usefulness of a duality is enriched if we have an axiomatisation of the dual category. For many of the examples given in Chapter 4, such an axiomatisation is given. Denote the topology-free reduct of $\underset{\sim}{\mathbf{M}}$ by $\underset{\sim}{\mathbf{M}}^\flat$. In the best of all worlds, the standard axiomatisation of $\mathbb{ISP}^+(\underset{\sim}{\mathbf{M}}^\flat)$ via universal Horn sentences also axiomatises $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$. When this holds, the discretely topologised structure $\underset{\sim}{\mathbf{M}}$ and the class $\mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ are said to be **standard**. This is true, for example, when $\underset{\sim}{\mathbf{M}}^\flat$ is a cyclic group or a two-element semilattice, but fails when $\underset{\sim}{\mathbf{M}}^\flat$ is a two-element ordered set. The theory of standardness began in [9] and has grown into an area of independent interest [8, 10, 28, 76, 77, 78, 75]. Standardness has turned out to have some surprising connections; for example, with constraint satisfaction problems [53], with residual bounds of topological algebras [52], with the general theory of Bohr compactifications [24], and with full dualities [34].

Examples of strong dualities in the case that $\underline{\mathbf{M}}$ is a total structure can be found in [17, 54].

Chapter 5: Sample Applications

Chapter 6: What Makes a Duality Useful?

Chapter 7: Piggyback Dualities

Piggybacking is a technique which, loosely speaking, says that if $\underline{\mathbf{M}}$ is dualisable, then an efficient way to find a structure $\underset{\sim}{\mathbf{M}}$ which yields a duality on $\mathcal{A} = \mathbb{ISP}(\underline{\mathbf{M}})$

is to ride piggyback on a known duality for an algebra $\underline{\mathbf{D}}$ which is closely related to $\underline{\mathbf{M}}$ by using the relationship between $\underline{\mathbf{M}}$ and $\underline{\mathbf{D}}$ to extract $\underset{\sim}{\mathbf{M}}$ from the structure $\underset{\sim}{\mathbf{D}}$ which dualises $\underline{\mathbf{D}}$. For example, $\underline{\mathbf{M}}$ might have an underlying bounded distributive lattice structure in which case we would choose $\underline{\mathbf{D}}$ to be the two-element bounded distributive lattice and we would try to obtain $\underset{\sim}{\mathbf{M}}$ from our knowledge of Priestley duality on the bounded distributive lattice underlying $\underline{\mathbf{M}}$.

Multisorted dualities. In the next section we shall develop a piggyback theorem for algebras with an underlying bounded distributive lattice. Since many of the applications of this result in the literature are to varieties rather than to quasi-varieties, we now give a brief overview of the modifications to our general theory which are required to extend it from the quasi-variety generated by $\underline{\mathbf{M}}$ to the variety generated by $\underline{\mathbf{M}}$.

Let $\underline{\mathfrak{M}}$ be a finite set of finite algebras of type F and let $\mathcal{A} := \mathbb{ISP}(\underline{\mathfrak{M}})$ be the quasi-variety generated by $\underline{\mathfrak{M}}$. For $\underline{\mathbf{M}}_1, \underline{\mathbf{M}}_2 \in \underline{\mathfrak{M}}$, we allow the possibility that $\underline{\mathbf{M}}_1$ and $\underline{\mathbf{M}}_2$ are isomorphic but, for notational convenience, we assume that the underlying sets M_1 and M_2 are disjoint. Although the finiteness assumptions on $\underline{\mathfrak{M}}$ can be relaxed (see Davey and Priestley [35] on which this and the next section are based), we shall not do so here as the stipulation that $\underline{\mathfrak{M}}$ is a finite set of finite algebras ensures that the results of the previous chapters can be lifted up to this more general setting.

The structure $\underset{\sim}{\mathbf{M}}$ must now be replaced with a **multisorted structure**

$$\underset{\sim}{\mathfrak{M}} = \langle \dot{\bigcup}\{ M \mid M \in \mathcal{M} \}; G^{\underset{\sim}{\mathfrak{M}}}, H^{\underset{\sim}{\mathfrak{M}}}, R^{\underset{\sim}{\mathfrak{M}}}, \mathcal{T}^{\underset{\sim}{\mathfrak{M}}} \rangle,$$

where

(i) $G^{\underset{\sim}{\mathfrak{M}}}$ consists of a homomorphism $g^{\underset{\sim}{\mathfrak{M}}} : \underline{\mathbf{M}}_1 \times \cdots \times \underline{\mathbf{M}}_n \to \underline{\mathbf{M}}_{n+1}$ (where $\underline{\mathbf{M}}_i \in \underline{\mathfrak{M}}$ for all i), for each n-ary operation symbol $g \in G$,

(ii) $H^{\underset{\sim}{\mathfrak{M}}}$ consists of a homomorphism $h^{\underset{\sim}{\mathfrak{M}}} : \mathbf{D} \to \underline{\mathbf{M}}_{n+1}$ (where $\mathbf{D}$ is a proper subalgebra of $\underline{\mathbf{M}}_1 \times \cdots \times \underline{\mathbf{M}}_n$ and $\underline{\mathbf{M}}_i \in \underline{\mathfrak{M}}$ for all i), for each n-ary partial operation symbol $h \in H$,

(iii) $R^{\underset{\sim}{\mathfrak{M}}}$ consists of an n-ary relation $r^{\underset{\sim}{\mathfrak{M}}}$ that forms a subalgebra of $\underline{\mathbf{M}}_1 \times \cdots \times \underline{\mathbf{M}}_n$ (where $\underline{\mathbf{M}}_i \in \underline{\mathfrak{M}}$ for all i), for each n-ary relation symbol $r \in R$, and

(iv) $\mathcal{T}^{\underset{\sim}{\mathfrak{M}}}$ is the discrete topology on $\dot{\bigcup}\{ M \mid M \in \mathcal{M} \}$.

By analogy with the case where $\underline{\mathfrak{M}}$ consists of a single algebra, we refer to the elements of $G^{\underset{\sim}{\mathfrak{M}}}$ as **(multisorted) operations**, to the elements of $H^{\underset{\sim}{\mathfrak{M}}}$ as **(multisorted) partial operations** and to the elements of $R^{\underset{\sim}{\mathfrak{M}}}$ as **(multisorted) relations** on $\mathcal{M}$ and we summarise (i)–(iii) by saying that the structure on $\underset{\sim}{\mathfrak{M}}$ is **algebraic over $\underline{\mathfrak{M}}$**.

Assume that $\langle X; \mathcal{T} \rangle$ is a topological space written as a disjoint union indexed by $\mathcal{M}$:

$$X = \dot{\bigcup}\{ X_M \mid M \in \mathcal{M} \}.$$

If, for every operation $g^{\underset{\sim}{\mathcal{M}}}\colon \underline{\mathbf{M}}_1 \times \cdots \times \underline{\mathbf{M}}_n \to \underline{\mathbf{M}}_{n+1}$ in $G^{\underset{\sim}{\mathcal{M}}}$, there is a corresponding map $g^{\mathbf{X}}\colon X_{M_1} \times \cdots \times X_{M_n} \to X_{M_{n+1}}$, and similarly for the partial operations in $H^{\underset{\sim}{\mathcal{M}}}$ and relations in $R^{\underset{\sim}{\mathcal{M}}}$, then we refer to $\mathbf{X} = \langle X; G^{\mathbf{X}}, H^{\mathbf{X}}, R^{\mathbf{X}}, \mathcal{T}\rangle$ as an $\mathcal{M}$-**sorted structure** of the same type as $\underset{\sim}{\mathcal{M}}$ and X_M is called the M-**sort** of $\mathbf{X}$. If $\mathbf{X}$ and $\mathbf{Y}$ are $\mathcal{M}$-sorted structures of the same type as $\underset{\sim}{\mathcal{M}}$, then a map $\varphi\colon X \to Y$ is a **morphism** provided it is continuous and preserves the sorts (that is, φ maps X_M into Y_M) and the (partial) operations and relations in the obvious sense. The concepts of **substructure**, **isomorphism**, **embedding** and **product** also have natural multisorted definitions. In particular, if S is a non-empty set, then the **power** $\underset{\sim}{\mathcal{M}}^S$ is the $\mathcal{M}$-sorted structure whose underlying set is

$$\mathcal{M}^S := \dot{\bigcup}\{\, M^S \mid M \in \mathcal{M}\,\}$$

with the obvious topology, and (partial) operations and relations extended pointwise in the natural manner.

As expected, we define $\mathcal{X}$ to be the category of all $\mathcal{M}$-sorted structures of the same type as $\underset{\sim}{\mathcal{M}}$ which are isomorphic to a closed substructure of some power $\underset{\sim}{\mathcal{M}}^S$ of $\underset{\sim}{\mathcal{M}}$ with $S \neq \varnothing$, in symbols, $\mathcal{X} = \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathcal{M}})$. It is straightforward to check that, for every $\mathbf{A} \in \mathcal{A}$,

$$D(\mathbf{A}) := \dot{\bigcup}\{\, \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}}) \mid M \in \mathcal{M}\,\}$$

is a closed substructure of $\underset{\sim}{\mathcal{M}}^A$, and for every $\mathbf{X} \in \mathcal{X}$, the homset $\mathcal{X}(\mathbf{X}, \underset{\sim}{\mathcal{M}})$ forms a subalgebra of $\prod\{\, \underline{\mathbf{M}}^{X_M} \mid M \in \mathcal{M}\,\}$. Hence, we obtain contravariant functors $D\colon \mathcal{A} \to \mathcal{X}$ and $E\colon \mathcal{X} \to \mathcal{A}$ which form a dual adjunction $\langle D, E, e, \varepsilon\rangle$ between $\mathcal{A}$ and $\mathcal{X}$. For all $\mathbf{A} \in \mathcal{A}$ and all $\mathbf{X} \in \mathcal{X}$, the morphisms

$$e_{\mathbf{A}}\colon \mathbf{A} \to ED(\mathbf{A}) \quad \text{and} \quad \varepsilon_{\mathbf{X}}\colon \mathbf{X} \to DE(\mathbf{X}),$$

which are given by evaluation, are embeddings. As in the single-sorted case, the dual of the algebra in $\mathcal{A}$ freely generated by a non-empty set S is isomorphic in $\mathcal{X}$ to $\underset{\sim}{\mathcal{M}}^S$.

Almost all of the results proved elsewhere in this text extend naturally and easily to the multisorted setting. An exception is the Two-For-One Strong Duality Theorem and results which depend upon it. Fortunately, the NU Duality Theorem and the NU Strong Duality Theorem do extend. It follows at once that any finitely generated variety of algebras which have an underlying lattice structure has a strong multisorted duality. In the next section we shall show that if the algebras have an underlying distributive lattice then the NU Duality Theorems may be bypassed and we can obtain a duality by riding piggyback on Priestley's duality for the underlying distributive lattices.

Piggyback dualities for distributive-lattice-based algebras. Most of the algebras which arise in algebraic logic are bounded distributive lattices with additional operations corresponding to implication or various types of negation.

In this section we shall present a theorem which applies to a finitely generated variety of algebras of this sort. In fact, the original theorem as proved in Davey and Priestley [35] applies to algebras based in any quasi-variety which itself has a multisorted duality. Since almost all applications in the literature have been to produce (multisorted) dualities for algebras based in bounded distributive lattices, it is this version which we present here.

To avoid excessive use of subscripts and superscripts, in this chapter we shall denote the two-element bounded distributive lattice by $\underline{\mathbf{D}} := \langle\{0,1\};\vee,\wedge,0,1\rangle$ and the two-element Priestley space by $\underset{\sim}{\mathbf{D}} := \langle\{0,1\};\leqslant,\mathcal{T}\rangle$. Priestley's duality tells us that $\underset{\sim}{\mathbf{D}}$ yields a strong duality between $\mathcal{D} := \mathbb{ISP}()D$, which now denotes the variety of *bounded* distributive lattices, and $\mathcal{P} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{D}})$, the category of Priestley spaces (see Exercise 4.5). Denote the hom-functors which yield this duality by $H\colon \mathcal{D} \to \mathcal{P}$ and $K\colon \mathcal{P} \to \mathcal{D}$ and, for each $\mathbf{A} \in \mathcal{D}$ and $\mathbf{Y} \in \mathcal{P}$, denote the corresponding evaluation maps by $k_{\mathbf{A}}\colon \mathbf{A} \to KH(\mathbf{A})$ and $\kappa_{\mathbf{Y}}\colon \mathbf{Y} \to HK(\mathbf{Y})$. We shall say that the class $\underline{\mathcal{M}}$ has a **term-reduct in $\mathcal{D}$** if there are binary terms $\vee$ and $\wedge$ and constant unary terms z and u of type F such that $\underline{\mathbf{M}}^\flat := \langle M;\vee^{\underline{\mathbf{M}}},\wedge^{\underline{\mathbf{M}}},0^{\underline{\mathbf{M}}},1^{\underline{\mathbf{M}}}\rangle$ is a bounded distributive lattice for each algebra $\underline{\mathbf{M}} \in \underline{\mathcal{M}}$, where $0^{\underline{\mathbf{M}}}$ and $1^{\underline{\mathbf{M}}}$ are the values in M of the constant unary term functions $z^{\underline{\mathbf{M}}}$ and $u^{\underline{\mathbf{M}}}$, respectively.

4.2.1 Piggyback Duality Theorem. *Assume that $\underline{\mathcal{M}}$ is a finite set of finite algebras which has a term-reduct in $\mathcal{D}$ and let $\mathcal{A} := \mathbb{ISP}(\underline{\mathcal{M}})$. For each $M \in \mathcal{M}$, let Ω_M be a subset of $\mathcal{D}(\underline{\mathbf{M}}^\flat,\underline{\mathbf{D}})$. Let $\underset{\sim}{\mathcal{M}} = \langle\dot{\bigcup}\{\,M \mid M \in \mathcal{M}\,\};G,R,\mathcal{T}\rangle$, where*

(i) *R is the set of all $\mathcal{A}$-subalgebras of $\underline{\mathbf{M}}_1 \times \underline{\mathbf{M}}_2$ which are maximal in*

$$(\omega_1,\omega_2)^{-1}(\leqslant) := \{\,(a,b) \in M_1 \times M_2 \mid \omega_1(a) \leqslant \omega_2(b)\},$$

where $\omega_1 \in \Omega_{M_1}$, $\omega_2 \in \Omega_{M_2}$ and $M_1, M_2 \in \mathcal{M}$,

(ii) *$G \subseteq \bigcup\{\,\mathcal{A}(\underline{\mathbf{M}}_1,\underline{\mathbf{M}}_2), \mid M_1, M_2 \in \mathcal{M}\,\}$ satisfies the separation condition*

(S) *for all $M_1 \in \mathcal{M}$ and all $a \neq b$ in M_1, we have $\omega(a) \neq \omega(b)$, for some $\omega \in \Omega_{M_1}$, or $\omega(g(a)) \neq \omega(g(b))$, where $\omega \in \Omega_{M_2}$ for some $M_2 \in \mathcal{M}$ and $g \in \mathcal{A}(\underline{\mathbf{M}}_1,\underline{\mathbf{M}}_2)$ is a composite of a finite number of maps from G,*

(iii) *$\mathcal{T}$ is the discrete topology.*

Then $\underset{\sim}{\mathcal{M}}$ yields a duality on $\mathcal{A}$.

The maps $\omega \in \Omega_M$ are referred to as the **carriers** of the piggyback duality, as they allow us to transport the structure from $\underset{\sim}{\mathbf{D}}$ up to the multisorted structure $\underset{\sim}{\mathcal{M}}$, while the relations in R are referred to as the **piggyback relations** on $\underline{\mathcal{M}}$.

Even in the single-sorted case, there is often more than one choice for the sets Ω_M and G which guarantee that the separation condition (S) holds. In practice, we often try to minimise the size of the sets Ω_M at the expense of increasing the size of the set G as this reduces the size of R. At the other end of the spectrum, $\Omega_M = \mathcal{D}(\underline{\mathbf{M}}^\flat,\underline{\mathbf{D}})$ for each $M \in \mathcal{M}$ and $G = \varnothing$.

7.2.3 Notation. To simplify our notation, from now on we shall discontinue the use of the $\mathbf{A}^\flat$ notation and will not distinguish notationally between the algebra $\mathbf{A}$ and its term-reduct in $\mathfrak{D}$. In any given situation it will be clear from the context which persona, $\mathbf{A}$ or $\mathbf{A}^\flat$, is involved.

Restricted Priestley dualities. Whenever we have a class $\mathcal{A}$ of algebras which have a term-reduct in the class $\mathfrak{D}$ of bounded distributive lattices, we can regard $\mathcal{A}$ as a subcategory of the category $\mathfrak{D}$ of bounded distributive lattices. Provided we can describe (up to isomorphism) the objects in the category $\mathcal{P}$ of Priestley spaces which are of the form $H(\mathbf{A})$ for $\mathbf{A} \in \mathcal{A}$ and can describe the morphisms in $\mathcal{P}$ which are of the form $H(u)$ for some $u \in \mathcal{A}(\mathbf{A}, \mathbf{B})$, then we may form the category $\mathcal{Y}$ consisting of all (isomorphic copies of) such objects and morphisms. Restricting the Priestley functors H and K to $\mathcal{A}$ and $\mathcal{Y}$ yields a dual equivalence between $\mathcal{A}$ and $\mathcal{Y}$. This is known as the **restricted Priestley duality** for $\mathcal{A}$.

In practice, it is often convenient to identify $H(\mathbf{A})$ with the set of prime filters of $\mathbf{A}$ and to identify $K(\mathbf{Y})$ with the set of clopen increasing subsets of $\mathbf{Y}$. The following result gathers together some particularly useful properties of the Priestley duality for $\mathfrak{D}$. Denote the order-theoretic duals of an ordered set $\mathbf{P} = \langle P; \leqslant \rangle$ and a bounded distributive lattice $\mathbf{L} = \langle L; \vee, \wedge, 0, 1\rangle$ by $\mathbf{P}^\partial$ and $\mathbf{L}^\partial$ respectively, that is, $\mathbf{P}^\partial = \langle P; \geqslant \rangle$ and $\mathbf{L}^\partial = \langle L; \wedge, \vee, 1, 0\rangle$.

7.4.1 Properties of Priestley Duality. *Let $\mathbf{A}$ and $\mathbf{B}$ be bounded distributive lattices, let $\mathbf{X}$, $\mathbf{X}_i$ and $\mathbf{Y}$ be Priestley spaces, let u be a bounded-distributive-lattice homomorphism and let φ and φ_i be Priestley space morphisms.*

(i) $H(\mathbf{A} \oplus \mathbf{1}) \cong \mathbf{1} \oplus H(\mathbf{A})$ *and* $K(\mathbf{1} \oplus \mathbf{X}) \cong K(\mathbf{X}) \oplus \mathbf{1}$.

(ii) $H(\mathbf{A}^\partial) \cong H(\mathbf{A})^\partial$ *and* $K(\mathbf{X}^\partial) \cong K(\mathbf{X})^\partial$.

(iii) $\mathrm{Con}(\mathbf{A}) \cong \mathcal{C}\ell(H(\mathbf{A}))^\partial$ *and* $\mathrm{Con}(K(\mathbf{X})) \cong \mathcal{C}\ell(\mathbf{X})^\partial$, *where* $\mathcal{C}\ell(\mathbf{X})$ *denotes the lattice of closed subsets of* $\mathbf{X}$.

(iv) (a) $H(\mathbf{A} \times \mathbf{B}) \cong H(\mathbf{A}) \,\dot\cup\, H(\mathbf{B})$, *and* (b) $K(\mathbf{X} \,\dot\cup\, \mathbf{Y}) \cong K(\mathbf{X}) \times K(\mathbf{Y})$.

(v) (a) *A homomorphism* $u \colon \mathbf{A} \to \mathbf{B}$ *is surjective (an embedding) if and only if the map* $H(u) \colon H(\mathbf{B}) \to H(\mathbf{A})$ *is an embedding (surjective).*

(b) *A morphism* $\varphi \colon \mathbf{X} \to \mathbf{Y}$ *is surjective (an embedding) if and only if the map* $K(\varphi) \colon H(\mathbf{Y}) \to H(\mathbf{X})$ *is an embedding (surjective).*

(vi) *If the images of the morphisms* $\varphi_i \colon \mathbf{X}_i \to \mathbf{Y}$, *for* $i \in I$, *are jointly dense in* $\mathbf{Y}$, *that is, if* $\bigcup_{i \in I} \varphi_i(X_i)$ *is dense in* $\mathbf{Y}$, *then the natural map* $\sqcap_{i\in I} K(\varphi_i) \colon K(\mathbf{Y}) \to \prod_{i \in I} K(\mathbf{X}_i)$ *is an embedding. In particular, if the morphisms* $\varphi_i \colon \mathbf{X}_i \to, \mathbf{Y}$, *for* $i \in I$, *are jointly surjective, then* $\sqcap_{i\in I} K(\varphi_i)$ *is an embedding.*

7.4.2 Ockham Algebras. An algebra $\mathbf{A} = \langle A; \vee, \wedge, f, 0, 1\rangle$ is an **Ockham algebra** if $\langle A; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and f corresponds to a negation operation which satisfies the de Morgan laws, that is,

$$f(0) = 1, \quad f(1) = 0, \quad f(a \vee b) = f(a) \wedge f(b) \quad \text{and} \quad f(a \wedge b) = f(a) \vee f(b),$$

for all $a, b \in A$. The variety of Ockham algebras is denoted by $\mathcal{O}$. The varieties of Stone algebras (see 4.3.6), of de Morgan algebras (see 4.3.15) and of Kleene algebras (see 4.3.9) are subvarieties of $\mathcal{O}$. There is an extensive literature on the variety of Ockham algebras and its subvarieties: see Blyth and Varlet [6] and the references given there. The restricted Priestley duality for Ockham algebras is particularly easy to describe.

7.4.3 Restricted Priestley Duality for $\mathcal{O}$. The operation f on an Ockham algebra $\mathbf{A}$ is nothing more than a dual endomorphism of the distributive lattice term-reduct of $\mathbf{A}$, that is, $f \in \mathcal{D}(\mathbf{A}, \mathbf{A}^{\partial})$. Hence, by (ii) of the proposition above, the operation f corresponds to a morphism $g \in \mathcal{P}(H(\mathbf{A}), H(\mathbf{A})^{\partial})$. Thus the objects of the restricted category $\mathcal{Y}_{\mathcal{O}}$, known as **Ockham spaces**, are of the form $\langle Y; g, \leqslant, \mathcal{T}\rangle$, where $\langle Y; \leqslant, \mathcal{T}\rangle$ is a Priestley space and $g \colon Y \to Y$ is a continuous, order-reversing map. Let $c(d) = d'$ be the usual Boolean complement on $\{0, 1\}$, then, for $\mathbf{A} \in \mathcal{O}$, the map g is defined on the set $H(\mathbf{A}) = \mathcal{D}(\mathbf{A}, \underline{\mathbf{D}})$ by $g(y) := c \circ y \circ f$ and, for $\mathbf{Y} \in \mathcal{Y}_{\mathcal{O}}$, the map f is defined on $K(\mathbf{Y}) = \mathcal{P}(\mathbf{Y}, \underset{\sim}{\mathbf{D}})$ by $f(\alpha) := c \circ \alpha \circ g$. The morphisms of $\mathcal{Y}_{\mathcal{O}}$ are continuous order-preserving maps which preserve g.
Extensive use has been made of the restricted Priestley duality for $\mathcal{O}$, for example, in Urquhart [79], where it was first written down, in Blyth and Varlet [6] and in Davey and Priestley [35].

In order to highlight the differences between natural dualities and restricted Priestley dualities, we shall compare them on the variety $\mathcal{K}$ of Kleene algebras. Recall from 4.3.9 that $\mathcal{K} = \mathbb{ISP}(\underline{\mathbf{K}})$ is the subvariety of $\mathcal{O}$ generated by the three-element Kleene algebra $\underline{\mathbf{K}}$. To find the restricted Priestley duality for $\mathcal{K}$, it remains to characterise Ockham spaces $\mathbf{Y}$ such that $K(\mathbf{Y})$ is a Kleene algebra. An alternative proof of the following result is given in Exercise 7.1.

Restricted Priestley Duality for $\mathcal{K}$. *Let $\mathbf{Y} = \langle Y; g, \leqslant, \mathcal{T}\rangle$ be an Ockham space. Then $\mathbf{Y}$ satisfies*

(K) $\quad (\forall y \in Y)\ g(g(y)) = y \quad$ *and* $\quad (\forall y \in Y)\ y \leqslant g(y)$ *or* $y \geqslant g(y)$.

if and only if $\mathbf{Y}$ is the Priestley dual of a Kleene algebra, that is, $K(\mathbf{Y})$ is a Kleene algebra.

The restricted Priestley dual category for $\mathcal{K}$ is therefore the full subcategory $\mathcal{Y}_{\mathcal{K}}$ of $\mathcal{Y}_{\mathcal{O}}$ consisting of the Ockham spaces which satisfy (K). It is an easy matter to construct (finite or infinite) Ockham spaces in $\mathcal{Y}_{\mathcal{K}}$.

7.4.6 Pros and Cons. We now have available to us both the natural duality (via 4.3.10) and the restricted Priestley duality (via 7.4.5). Which should we use and when? The restricted Priestley duality is a powerful tool in the study of individual Kleene algebras, particularly if we are concerned with the underlying distributive lattice. But for the study of more global properties of the class $\mathcal{K}$, the natural duality has many strengths over the restricted Priestley duality. Let $\mathcal{X}$ be the natural dual category as described in 4.3.10.

(1) Products in the natural dual $\mathcal{X}$ are simply the usual Cartesian products with structure defined pointwise from the factors. Although products exist in $\mathcal{Y}_{\mathcal{K}}$ (since coproducts exist in $\mathcal{K}$), they need not be Cartesian: the Cartesian product of two spaces satisfying (K) need not satisfy (K) !
(2) Free objects exist in $\mathcal{X}$ (see Exercise 3.5) while this fails in $\mathcal{Y}_{\mathcal{K}}$ (see Exercise 7.2).
(3) The dual in $\mathcal{X}$ of the free Kleene algebra $\mathbf{F}_{\underline{\mathbf{K}}}(S)$ is simply $\underset{\sim}{\mathbf{K}}^S$. It is not at all clear how to find the dual in $\mathcal{Y}_{\mathcal{K}}$ of $\mathbf{F}_{\underline{\mathbf{K}}}(S)$!

7.4.7 Working Together. Of course, we could ask for the best of both worlds. Since the categories $\mathcal{X}$ and $\mathcal{Y}_{\mathcal{K}}$ are both dually equivalent to $\mathcal{K}$, they are equivalent to one another. If we could give an explicit description of the functors between $\mathcal{X}$ and $\mathcal{Y}_{\mathcal{K}}$ which yield this equivalence, we would have a **translation process** between $\mathcal{X}$ and $\mathcal{Y}_{\mathcal{K}}$ permitting us to use the two dualities in tandem. We could then, for example, find (the Priestley dual of) the underlying distributive lattice of $\mathbf{F}_{\underline{\mathbf{K}}}(S)$ by applying the translation process to the object $\underset{\sim}{\mathbf{K}}^S$ in $\mathcal{X}$. This translation process has been worked out in a number of cases: for example, in the next section we shall give the description, from Davey and Priestley [35], of the translation between the restricted Priestley duality and a multisorted natural duality for the variety of Ockham algebras generated by a finite subdirectly irreducible algebra $\underline{\mathbf{M}}$, and the translation process in the case where $\underline{\mathbf{M}}$ is an n-valued Łukasiewicz algebra is described in Priestley [72].

It is often the case that the restricted Priestley duality for the quasi-variety generated by $\underline{\mathbf{M}}$ or, more generally, by $\underline{\mathcal{M}}$ is known and we want to apply the Piggyback Duality Theorem 7.2.1 to obtain a natural duality. Once we have chosen sets G and Ω_M, for $\underline{\mathbf{M}} \in \underline{\mathcal{M}}$, which satisfy the separation condition (S) of the Piggyback Duality Theorem, two questions need to be addessed.

(i) How can we conveniently describe the piggyback relations on $\underline{\mathcal{M}}$?
(ii) How do we find a 'small' subset R' of R such that $G \cup R'$ entails R (and therefore $\underset{\sim}{\mathcal{M}}' = \langle \dot{\bigcup}\{ M \mid M \in \mathcal{M} \}; G, R', \mathcal{T} \rangle$ yields a duality on the quasi-variety generated by $\underline{\mathcal{M}}$)?

In some cases we can use algebraic techniques as we did in the case of Heyting algebras in Lemma 7.3.1. Alternatively, we can attempt to answer these questions by riding piggyback on the *known* restricted Priestley duality. This technique was first used to answer question (i) in Davey and Priestley ([35] and [37]), and to answer question (ii) in Davey and Priestley [38]. We illustrate the technique in Sections 5 and 6 below.

A very useful $\mathfrak{D}$–$\mathfrak{P}$ 'dictionary', which describes the restricted Priestley dual for a number of classes, is given at the end of Priestley [71]. A list of 239 references which 'are related to, develop, and/or apply Priestley or natural dualities' is given by Adams and Dziobiak at the end of the *Studia Logica* Special Issue on Priestley Duality [2].

Piggyback dualities for Kleene algebras. (The remainder of Section 6 of Chapter 7 develops a two-sorted natural duality of each variety of Ockham algebras generated by a finite subdirectly irreducible algebra. When applied to Kleene algebras this is easily seen to be equivalent to the single-sorted natural duality given in Chapter 4. The advantage of the two-sorted duality is that it makes it very easy to describe the translation process between the natural duality and the restricted Priestley duality. We refer to pages 207–216 of the text for details.)

Update 5

A careful analysis of the Piggyback Duality Theorem, in the most general setting where $\underline{\mathbf{M}}$ is a possibly infinite structure, has been carried out in [23]. It is used to give a very short proof of the strong duality for Ockham algebras based on an infinite compact topological Ockham algebra whose underlying topological space is $\{0,1\}^{\mathbb{N}}$ with the product topology, which was originally proved in [44, 45]. It is also applied to establish strong dualities in the case that $\underline{\mathbf{M}}$ is a compact topological semilattice-based algebra.

The **Completely Dualisable Quasivariety Problem**, stated in this chapter, was solved in [11].

Chapter 8: Optimal Dualities and Entailment

Recall that $\underset{\sim}{\mathbf{M}}$ yields an **optimal duality** on $\mathcal{A}$ if $\underset{\sim}{\mathbf{M}}$ yields a duality but as soon as any relation or (partial) operation is eliminated from the structure on $\underset{\sim}{\mathbf{M}}$ the resulting structure $\underset{\sim}{\mathbf{M}}^*$ no longer yields a duality. In this chapter we will exhibit a systematic method for reducing any dualising structure of finite type to one which yields an optimal duality on $\mathcal{A}$.

According to the Duality and Entailment Theorem 2.4.3, a dualising set $G \cup H \cup R$ yields an optimal duality if and only if no member of $G \cup H \cup R$ is entailed by the rest of $G \cup H \cup R$. Our quest for optimal dualities begins with a simple question.

> Is there a finite algorithm to determine, for an algebraic relation s and finite sets G, H and R, whether or not $G \cup H \cup R$ entails s?

Notice that the definition of entailment does not provide such an algorithm since it asks that $G \cup H \cup R$ entail s on $D(\mathbf{A})$ for *every* $\mathbf{A} \in \mathcal{A}$. We will obtain a finite algorithm to determine entailment by establishing the startling fact that it is always sufficient to consider only the single finite algebra $\mathbf{A} = \mathbf{s}$! Capitalising on this observation, we will obtain a method to compute from a given finite dualising set all of its subsets which yield an optimal duality. This method will be fully illustrated in the cases of semi-primal algebras and Kleene algebras. This chapter is based on Davey and Priestley [39].

Test algebras—schizophrenia strikes again! In this section we show that the question of deciding whether or not a finite set $G \cup H \cup R$ entails a relation s is answerable by a finite algorithm.

The secret is a simple but lovely application of schizophrenia. Since the n-ary *relation* s is algebraic we may consider its pointwise extension $s^{D(\mathbf{s})}$ to the dual $D(\mathbf{s})$ of the *algebra* $\mathbf{s} \leqslant \underline{\mathbf{M}}^n$. We shall see that $G \cup H \cup R \vdash s$ if and only if $G \cup H \cup R$ entails s on $D(\mathbf{s})$. For this reason, we refer to the algebra $\mathbf{s}$ corresponding to an algebraic relation $s \in \mathcal{B}$ as a **test algebra**. Since $D(\mathbf{s})$ and M are finite, there is only a finite number of maps from $D(\mathbf{s})$ to M and consequently whether or not $G \cup H \cup R$ entails s is finitely determined provided $G \cup H \cup R$ is finite.

8.1.3 Test Algebra Lemma. *Let $G \cup H \subseteq \mathcal{P}$, let $R \subseteq \mathcal{B}$ and let $s \in \mathcal{B}$. Then the following are equivalent:*

(i) *$G \cup H \cup R$ entails s on each hom-closed subset of every power of M;*
(ii) *$G \cup H \cup R$ entails s;*
(iii) *$G \cup H \cup R$ entails s on $D(\mathbf{s})$.*

Notice that if G, H and R are all finite, then the verification of (iii) above is a finite process which provides the promised algorithm to test entailment. We close this section with a simple but important consequence of the Test Algebra Lemma along with an illustration of its usefulness.

8.1.4 Corollary. *In order to prove that $G \cup H \cup R$ entails s it suffices to prove that $G \cup H \cup R$ yields a duality on some isomorphic copy of the test algebra $\mathbf{s}$.*

8.2.3 Failsets. Let $\Omega \subseteq \mathcal{B}$, let $s \in \Omega$ and let $\gamma \colon D(\mathbf{s}) \to M$ be any map. Define

$$U = \mathrm{Fail}_{\mathbf{s}}(\gamma) := \{\, r \in \Omega \mid \gamma \text{ fails to preserve } r \,\}.$$

If $U \neq \varnothing$ we call U a **weak failset of** s (within Ω), and if $s \in U$ we call U a **failset of** s (within Ω). We refer to U as a **failset** if it is a failset of some $s \in \Omega$. For a fixed $s \in \Omega$, let

$$\mathscr{F}_s := \{\, \mathrm{Fail}_{\mathbf{s}}(\gamma) \mid \gamma \colon D(\mathbf{s}) \to M \text{ fails to preserve } s \,\}$$

be the family of all failsets of s and let $\mathscr{F} := \bigcup\{\, \mathscr{F}_s \mid s \in \Omega \,\}$ be the family of all failsets.

If γ is an evaluation map, say $\gamma = e_{\mathbf{s}}(c)$ for some $c \in s$, then the set $U = \mathrm{Fail}_{\mathbf{s}}(\gamma)$ is empty. Conversely, if Ω yields a duality on $\mathbf{s}$ and $U = \varnothing$, then γ is an evaluation map. In general, the size of U gives a measure of how far γ is from being an evaluation map.

Even when Ω is infinite, $\mathscr{F}_s$ contains an abundance of minimal elements. In fact these minimal failsets provide the key to understanding optimal dualities. To see how this works, we order both $\mathscr{F}_s$ and $\mathscr{F}$ by set inclusion. If U is a minimal element of $\mathscr{F}_s$ we call U a **locally minimal failset of** s and if U is minimal in $\mathscr{F}$ then we refer to U as a **globally minimal failset**. Clearly every globally minimal failset is a locally minimal failset, but not conversely. Since the size of $\mathrm{Fail}_s(\gamma)$ is a measure of how far γ is from being an evaluation, we should expect minimal failsets to play an important role in our theory as each one comes from a map γ which is as close as possible to an evaluation map without actually being one.

Another notion will help us make this connection. Let $\mathscr{S}$ be a family of non-empty sets and let T be a subset of their union. Then T is called a **transversal** of $\mathscr{S}$ if T intersects each $U \in \mathscr{S}$ but no proper subset of T does. If $\mathscr{S}$ is finite, or the sets in $\mathscr{S}$ are pairwise disjoint, then a transversal of $\mathscr{S}$ certainly exists. We now show how these ideas relate to optimal dualities.

8.3.1 Optimal Duality by Failsets Theorem. *Suppose that Ω yields a duality on $\mathcal{A}$ and $R \subseteq \Omega$.*

(i) *R yields a duality on $\mathcal{A}$ if and only if it intersects every failset.*

(ii) *R yields an optimal duality on $\mathcal{A}$ if and only if it is a transversal of the family of failsets.*

Moreover, if every failset contains a globally minimal failset—as is the case if Ω is finite—then

(iii) *R yields an optimal duality on $\mathcal{A}$ if and only if it is a transversal of the globally minimal failsets.*

We now turn our attention to globally minimal failsets, the transversals of which give us optimal dualities provided that Ω is a finite dualising set. They correspond to the coatoms of the lattice $\mathbf{\Lambda}$ of entailment closed subsets of Ω.

8.3.8 Globally Minimal Failset Theorem. *Let $\varnothing \neq U \subseteq \Omega$. Then the following are equivalent:*

(i) *$\Omega \backslash U$ is a coatom in the lattice $\mathbf{\Lambda}$;*

(ii) *U is a globally minimal failset;*

(iii) *U is a locally minimal failset of r for all $r \in U$;*

(iv) *U is a (weak) failset and $(\Omega \backslash U) \cup \{s\} \vdash r$, for all $r, s \in U$.*

Assume that Ω yields a duality on $\mathcal{A}$. We say that a subset U of Ω is **unavoidable** (within Ω) if whenever a subset R of Ω yields a duality on $\mathcal{A}$, then $R \cap U \neq \varnothing$. Notice that U is unavoidable if and only if $\Omega \backslash U$ does not yield a duality on $\mathcal{A}$.

The unavoidable subsets form an ordered set (under set inclusion) with Ω as its top. A subset U of Ω will be called a **minimal unavoidable set** (within Ω) if it is a minimal element of this ordered set.

8.3.10 Optimal Duality Theorem. *Assume that $\Omega \subseteq \mathcal{B}$ is finite and yields a duality on $\mathcal{A}$ and let $R \subseteq \Omega$. Then the following are equivalent:*

(i) *R yields an optimal duality on $\mathcal{A}$;*
(ii) *R is a transversal of the globally minimal failsets within Ω;*
(iii) *R is a transversal of the complements of coatoms in the lattice $\mathbf{\Lambda}$ of entailment closed subsets of Ω;*
(iv) *R is a transversal of the minimal unavoidable sets within Ω.*

In almost all cases described in Chapter 4 we found that the dualising relations were either unary or binary. When $\Omega \subseteq \mathbb{S}(\underline{\mathbf{M}}) \cup \mathbb{S}(\underline{\mathbf{M}}^2)$ yields a duality on $\mathcal{A}$ there is a natural concept of absolutely unavoidable relation. If $s \in \mathbb{S}(\underline{\mathbf{M}})$ we define $s^{\smile}$ to be s. We say that $s \in \Omega \subseteq \mathbb{S}(\underline{\mathbf{M}}) \cup \mathbb{S}(\underline{\mathbf{M}}^2)$ is **absolutely unavoidable** (within Ω) if $\{s, s^{\smile}\}$ is a (necessarily minimal) unavoidable set.

Kleene algebras once again. In 4.3.9 and 4.3.10 in Chapter 4 we introduced the class, $\mathcal{K}$, of Kleene algebras and applied the NU Strong Duality Corollary 3.3.9 to obtain a strong duality for this class. The schizophrenic object is

$$\underline{\mathbf{K}} = \langle \{0, a, 1\}; \vee, \wedge, \neg, 0, 1 \rangle \quad \text{and} \quad \underset{\sim}{\mathbf{K}} = \langle \{0, a, 1\}; \preccurlyeq, \sim, K_0, \mathcal{T} \rangle$$

where

$$0 < a < 1, \ \neg 0 = 1, \ \neg 1 = 0 \quad \text{and} \ \ \neg a = a,$$

and $\preccurlyeq$ is the order with $0 \prec a$ and $1 \prec a$ while $\sim \, = K^2 \backslash \{(0,1),(1,0)\}$ and $K_0 = \{0, 1\}$. We shall now see that not only is this duality optimal, but in a very natural sense it is the unique duality using algebraic relations of minimal arity. Both $\preccurlyeq$ and $\sim$ turn out to be absolutely unavoidable in $\Omega = \mathbb{S}(\underline{\mathbf{K}}) \cup \mathbb{S}(\underline{\mathbf{K}}^2)$ while K_0 is contained in a unique globally minimal failset U and is the only unary relation in U.

8.4.1 The Duality Again. We begin by proving again that $\underset{\sim}{\mathbf{K}}$ yields a duality on $\mathcal{K}$. This time round we shall try to emphasise the choices which are available along the way. By the NU Duality Theorem 2.3.4 the set $\Omega := \mathbb{S}(\underline{\mathbf{K}}) \cup \mathbb{S}(\underline{\mathbf{K}}^2)$ yields a duality on $\mathcal{K}$. As in the proof of 4.3.10, it is easy to show by hand that $\underline{\mathbf{K}}^2$ has 11 subalgebras:

$$\Delta_K, \ \Delta_{K_0}, \ \preccurlyeq, \ \succcurlyeq := \preccurlyeq^{\smile}, \ \sim, \ K \times K, \ K \times K_0, \ K_0 \times K,$$
$$K_0 \times K_0, \ \preccurlyeq \cap \, (K_0 \times K) \ \text{ and } \ \succcurlyeq \cap \, (K \times K_0).$$

The lattice of subalgebras of $\underline{\mathbf{K}}^2$ is as shown in Figure 8.1: the only unlabelled relations are $\preccurlyeq \cap \, (K_0 \times K)$ and $\succcurlyeq \cap \, (K \times K_0)$. The meet-irreducible elements of $\mathbb{S}(\underline{\mathbf{K}}^2)$ are $K \times K_0$, $K_0 \times K$, $\preccurlyeq$, $\succcurlyeq$, and $\sim$. Any set which entails these relations will

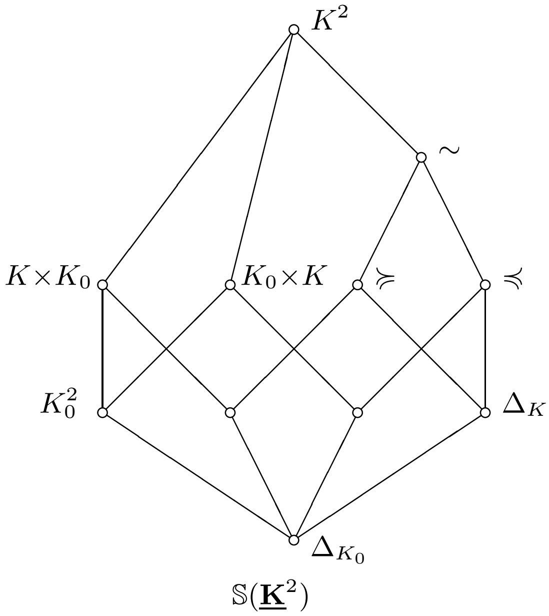

FIGURE 8.1 the subalgebras of $\underline{\mathbf{K}}^2$

entail their intersections, and hence entail all subalgebras of $\underline{\mathbf{K}}^2$. Consequently $R := \{\preccurlyeq, \sim, K_0\}$ entails all subalgebras of $\underline{\mathbf{K}}^2$ and so yields a duality on $\mathcal{K}$.

How can we modify R without destroying the duality? It is easily seen that each member of the set

$$U^{K_0} := \{K_0, \Delta_{K_0}, K_0^2, K_0 \times K, K \times K_0, \preccurlyeq \cap (K_0 \times K), \succcurlyeq \cap (K \times K_0)\}$$

entails K_0. Thus, we could replace K_0 with any other member of U^{K_0}, but this would have the disadvantage of replacing a unary relation with a binary one. The only other change which comes to mind would be the trivial one of replacing $\preccurlyeq$ with its converse $\succcurlyeq$. This certainly *feels* like an optimal duality. To see that it is we need to find the minimal unavoidable sets, or equivalently, the global minimal failsets within $\Omega := \mathbb{S}(\underline{\mathbf{K}}) \cup \mathbb{S}(\underline{\mathbf{K}}^2)$.

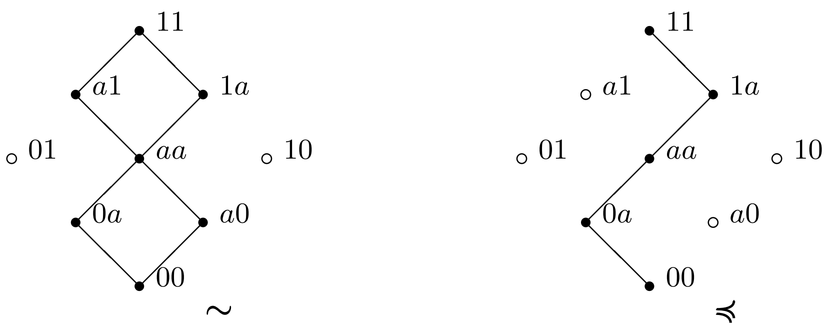

FIGURE 8.2 the subalgebras $\sim$ and $\preccurlyeq$ of $\underline{\mathbf{K}}^2$

8.4.2 The Globally Minimal Failsets. First, consider the relation $\sim$: see Figure 8.2. Let $x\colon \boldsymbol{\sim} \to \underline{\mathbf{K}}$ be a homomorphism. Because the fixpoint (a,a) of the Kleene negation must map to a, it is very easy to show that x is a projection. Thus $D(\boldsymbol{\sim}) = \{\rho_1, \rho_2\}$. Define $\gamma\colon D(\boldsymbol{\sim}) \to K$ by $\gamma(\rho_1) = 0$ and $\gamma(\rho_2) = 1$. Since $(0,1) \notin \sim$, and, by Lemma 8.1.1, $(\rho_1, \rho_2) \in \sim^{D(\sim)}$, we conclude that $\sim\, \in \mathrm{Fail}_{\sim}(\gamma)$, that is, $\mathrm{Fail}_{\sim}(\gamma)$ is a failset of $\sim$. We shall now show that $\mathrm{Fail}_{\sim}(\gamma) = \{\sim\}$. Note that $K_0^{D(\sim)} = \varnothing$ since, for $i = 1, 2$, we have $\rho_i((a,a)) = a \notin K_0$. Thus γ preserves K_0 and consequently $\mathrm{Fail}_{\sim}(\gamma)$ consists of binary relations. Let $r \in \mathrm{Fail}_{\sim}(\gamma)$. Then we can find $x, y \in D(\boldsymbol{\sim})$ such that $(x,y) \in r^{D(\sim)}$ and $(\gamma(x), \gamma(y)) \notin r$. As $(0,0), (1,1) \in r$, we must have $x \neq y$. Consequently, as r is closed under the Kleene negation, $(0,1), (1,0) \notin r$, that is, $r \subseteq \sim$. By the **r**-on-**s** Lemma 8.1.2, there is a homomorphism, namely $\varphi = \rho_1 \sqcap \rho_2$ or $\varphi = \rho_2 \sqcap \rho_1$, from $\boldsymbol{\sim}$ to $\mathbf{r}$. In either case, φ is one-to-one and hence $\varphi(\sim) \subseteq r \subseteq \sim$ implies that $r = \sim$. We have proved that $\mathrm{Fail}_{\sim}(\gamma) = \{\sim\}$, as claimed, and this failset is clearly minimal. Since this is also a minimal unavoidable set, $\sim$ is absolutely unavoidable.

(As subsets of K^2, the relation $\sim$ and the relational product $\succcurlyeq \cdot \preccurlyeq$ coincide. This identification fails on arbitrary spaces $D(\mathbf{A})$ for $\mathbf{A} \in \mathcal{K}$. Indeed, in a typically schizophrenic way, the identification fails on the dual $D(\boldsymbol{\sim})$ of the corresponding test algebra, that is, as we shall verify in 9.1.3,

$$\succcurlyeq^{D(\sim)} \cdot \preccurlyeq^{D(\sim)} \neq (\succcurlyeq \cdot \preccurlyeq)^{D(\sim)}.$$

Since γ preserves both $\succcurlyeq$ and $\preccurlyeq$, but does not preserve $\sim$, this shows that, in general, relational product cannot be an admissible construct.)

An almost identical argument applies to the relation $\preccurlyeq$. We leave it to the reader to check that $D(\preccurlyeq) = \{\rho_1, \rho_2\}$ and that, with γ defined by $\gamma(\rho_1) = a$ and $\gamma(\rho_2) = 0$, we have $\mathrm{Fail}_{\preccurlyeq}(\gamma) = \{\preccurlyeq, \succcurlyeq\}$, whence this failset is clearly minimal and $\preccurlyeq$ is also absolutely unavoidable within Ω.

Notice that any failset intersecting either $\{\preccurlyeq, \succcurlyeq\}$ or $\{\sim\}$ must contain that failset, and that the absolutely avoidable relations K, Δ_K and K^2 belong to no failset. Thus any other minimal failset must be a subset of $\Omega \backslash \{\preccurlyeq, \succcurlyeq, \sim, K, \Delta_K, K^2\} = U^{K_0}$. Thus we can conclude our search for the minimal unavoidable sets within Ω by showing that U^{K_0} itself is a minimal unavoidable set. First we prove that U^{K_0} is a failset of K_0. The dual of $\mathbf{K}_0$ is simply $D(\mathbf{K}_0) = \{\rho_1\}$, where $\rho_1 \colon \mathbf{K}_0 \to \underline{\mathbf{K}}$ is the inclusion map. Define $\gamma\colon D(\mathbf{K}_0) \to K$ by $\gamma(\rho_1) = a$. Again, the reader should show that the only unary relation in $\mathrm{Fail}_{\mathbf{K}_0}(\gamma)$ is K_0 and that a binary relation r is in $\mathrm{Fail}_{\mathbf{K}_0}(\gamma)$ if and only if $(a,a) \notin r$. It follows that $\mathrm{Fail}_{\mathbf{K}_0}(\gamma) = U^{K_0}$. It is easy to check by hand, using the Constructs for Entailment 2.4.5, that $\{\preccurlyeq, s\} \vdash r$ for all $r, s \in U^{K_0}$, whence U^{K_0} is a globally minimal failset by the equivalence of (ii) and (iv) in the Globally Minimal Failset Theorem 8.3.8.

This shows that there are exactly three minimal failsets within Ω, namely

$$\{\preccurlyeq, \succcurlyeq\}, \quad \{\sim\} \quad \text{and} \quad U^{K_0}.$$

By the Optimal Duality Theorem 8.3.10, the subsets of Ω which yield optimal dualities on $\mathcal{K}$ are the transversals of this family. Therefore, as expected, the only choices available are whether to use $\preccurlyeq$ or its converse and which member of U^{K_0} to use. Thus, as claimed, there is an essentially unique optimal duality for $\mathcal{K}$ using relations of minimal arity, namely $R = \{\preccurlyeq, \sim, K_0\}$.

Chapter 9: Completeness Theorems for Entailment

Update 6

Chapter 9 contains a proof that the constructs listed in Constructs for Entailment 2.4.5, when augmented with one further construct, are complete. A stronger form of entailment, known as **structural entailment** is introduced and characterised in [26]. A detailled analysis of different forms of entailment relevant to duality theory is conducted in [33].

Chapter 10: Dualisable Algebras

Update 7

This chapter contains the Inherent Non-dualisable Algebra Theorem and its proof. The weaker, but more widely applicable, Non-dualisability Lemma is stated and proved in [66, 3.4.1]. The chapter contains six problems, three of which have been solved. The **Countability Problem** and the **Finite Degree Problem** were solved in the negative in [65]. The **Inherent Non-dualisability Problem** was solved in [11] by proving that every finite unary algebra has a dualisable extension. Part of the Inherent Non-dualisability Problem asks whether dualisabilty is independent of the generator, i.e., if $\mathbb{ISP}(\underline{\mathbf{M}}_1) = \mathbb{ISP}(\underline{\mathbf{M}}_2)$, for finite algebras $\underline{\mathbf{M}}_1$ and $\underline{\mathbf{M}}_2$, and $\underline{\mathbf{M}}_1$ is dualisable, does it follow that $\underline{\mathbf{M}}_2$ is also dualisable? This was answered in the affirmative in [43] and [73] (independently). Subsequently, corresponding independence-of-the-generator results have been proved for both strong dualisability [48] and full dualisability [21].

Of the remaining three problems, the **Dualisibility Problem** and the **Dualisable Clones Problem** are essentially equivalent and ask which finite algebras are dualisable. These problems are probably unanswerable. Nevertheless, significant progress has been made within restricted classes of algebras; for example, amongst semigroups (and groups) [30, 30, 46, 50, 62, 1, 51, 58] and semilattice-based algebras [29, 32, 12]. The NU Duality Theorem has been extended to a theorem that applies to both finite lattice-based algebras and finite group-based algebras [58].

Dualisability appears to be a finiteness condition on the quasivariety generated by a finite algebra. Its connection to other familiar finiteness conditions remains unclear. A connection between the residual character of the variety generated by a finite algebra and its dualisability status has been revealed [32, 58]. While we have an example of an infinite dualisable algebra with no finite base for its equations [29], no finite example is known—see [27, 59, 5, 61].

The remaining problem, the **Decidability Problem**, is the holy grail of the theory of natural dualities and asks if the dualisability of a finite algebra of finite type is decidable. It remains unsolved. The second part of the problem, which asks if the existence of a near-unanimity term on a finite algebra is decidable, was answered in the affirmative in [60].

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