The FEP for some noncommutative varieties of fully-distributive residuated lattices

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1 Introduction

A class of algebras $\mathcal{K}$ is said to have the *finite embeddability property* (FEP), if for every algebra $\mathbf{A}$ in $\mathcal{K}$ and every *finite* partial subalgebra $\mathbf{B}$ of $\mathbf{A}$, there exists a finite algebra $\mathbf{D}$ in $\mathcal{K}$ such that $\mathbf{B}$ embeds into $\mathbf{D}$. Recall that $\mathbf{B}$ is a *partial* subalgebra of $\mathbf{A}$, if B is a subset of A together with partial operations $f^{\mathbf{B}}$ for each n-ary operation $f^{\mathbf{A}}$ on A, where $f^{\mathbf{B}}$ is:

$$f^{\mathbf{B}}(b_1,\dots,b_n) = \begin{cases} f^{\mathbf{A}}(b_1,\dots,b_n), & \text{if } f^{\mathbf{A}}(b_1,\dots,b_n) \in \mathbf{B}. \\ \text{undefined}, & \text{if } f^{\mathbf{A}}(b_1,\dots,b_n) \notin \mathbf{B}. \end{cases}$$

The FEP is a strong property, as it yields decidability for finitely axiomatizable classes and generation by finite algebras for (quasi)varieties.

A residuated lattice is an algebra $(A, \wedge, \vee, \cdot, \backslash, /, 1)$ where $(A, \cdot, 1)$ is a monoid, $(A, \wedge, \vee)$ is a lattice and for all $a, b, c \in A$, we have $ab \leq c$ iff $a \leq c/b$ iff $b \leq a\backslash c$. As usual, we write $x \leq y$ for $x = x \wedge y$. It is not hard to see that the class of residuated lattices is a variety. For more on residuated lattices, see for example [5].

The FEP was studied for various classes of residuated lattices by W. Blok and C. van Alten in a series of papers. Since residuated lattices form algebraic semantics for substructural logics (see [5]), the FEP for a variety of residuated lattices yields the strong finite model property for the corresponding substructural logic. In that respect the FEP is a very desirable, but also fairly rare property.

In [4], among other things, the FEP is established for all subvarieties of integral (satisfying $x \leq 1$) residuated lattices axiomatized by equations over the language of join, multiplication and 1; the method used is that of *residuated frames*. In that respect integrality is a strong condition, but already in [7] it is replaced by the weaker condition

$$x^m \leq x^n \text{ for } m \neq n,\ m \geq 1,\ n \geq 0,$$

known as a *knotted inequality*; the price to pay for such a generalization is to assume commutativity (of multiplication). The variety of all residuated lattices satisfying a knotted inequality does not have the FEP, but in [1] it is shown that the FEP holds for an infinite collection of non-commutative varieties satisfying a knotted inequality. (Further subvarieties of these axiomatized by equations of over the language $\{\vee, \cdot, 1\}$ have the FEP.) Each of them is axiomatized by a monoid identity, the simplest of which is $xyx = xxy$, and in general it is given relative to a vector $a = (a_0, a_1, \ldots, a_r)$ of natural numbers whose sum is $r + 1$ and whose product is 0 (namely an additive, non-trivial, decomposition of the number $r + 1$):

$$xy_1xy_2 \cdots y_rx = x^{a_0}y_1x^{a_1}y_2 \cdots y_rx^{a_r}. \qquad (a)$$

In [3] it is shown, by developing a theory for a distributive version of residuated frames, that in the presence of integrality we can obtain the FEP for all varieties of residuated lattices that are distributive and are axiomatized over the language $\{\vee, \cdot, 1\}$ (actually, even further, we can allow the $\wedge$ connective in many places). For example, the FEP is established for all integral and *fully distributive* residuated lattices. In all residuated lattices multiplication distributes over join, but if, further, both multiplication and join distribute over meet, we call the residuated lattice fully distributive. Algebras such as lattice-ordered groups, Heyting algebras, and all semilinear residuated lattices (including MV-algebras and BL-algebras), are fully distributive residuated lattices. Furthermore, fully distributive residuated lattices admit a nice representation theorem [2].

In this submission, we relax the integrality condition to a combination of a knotted inequality (for $m > n$) and an equation (a), for some decomposition a of a positive integer, thus obtaining infinitely many varieties of fully distributive residuated lattices with the FEP, outside the setting of integrality or commutativity.

2 The construction of D

We consider a variety $\mathcal{V}$ of fully distributive residuated lattices axiomatixed by a knotted inequality $x^m \leq x^n$, $m > n$, and an equation of the form (a). (We may also assume that the axiomatization of the variety contains further equations over the language $\{\vee, \cdot, 1\}$, and even some controlled occurences of $\wedge$, as explained in [3].) We will show that $\mathcal{V}$ has the FEP.

We consider an algebra $\mathbf{A}$ in $\mathcal{V}$ and a finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$; let $B = \{b_1, b_2, \ldots, b_k\}$. Let $\mathbf{W} = (W, \circ, \curlywedge, \varepsilon)$ be the $\{\cdot, \wedge, 1\}$-subalgebra of $\mathbf{A}$ generated by B (note that we use different notation for the restrictions of the operations of $\mathbf{A}$ on W). Observe that polynomials over $(W, \circ, \curlywedge, \varepsilon)$ that contain a single variable x with a single occurrence must look like $u(x) = (y \circ x \circ z) \curlywedge w$ for $y, z, w \in W$, and since multiplication distributes over meet, we can even assume that y and z do not have $\curlywedge$ in them. We denote the set of all such polynomials by S_W and we define the set $W' = S_W \times B$, as well as the relation N from W to W', given by $xN(u, b)$ iff $u^{\mathbf{A}}(x) \leq^{\mathbf{A}} b$.For $X \subseteq W$ and $Y \subseteq W'$, we define $X^{\rhd} = \{z \in W' : (\forall x \in X)(x \ N \ z)\}$ and $Y^{\lhd} = \{w \in W : (\forall y \in Y)(w \ N \ y)\}$ and also the map γ_N on W by $\gamma_N(X) = X^{\rhd\lhd}$. We denote by $\gamma_N[\mathcal{P}(W)]$ the image of this map and call its members closed sets. The algebra

$\mathbf{W}^+_{\mathbf{A},\mathbf{B}} = (\gamma_{\mathbf{N}}[\wp(\mathbf{W})], \cap, \cup_{\mathbf{N}}, \circ_{\mathbf{N}}, \backslash, /)$ is called the *Galois algebra* of $\mathbf{W}_{\mathbf{A},\mathbf{B}}$, where for $X, Y \subseteq W$ we define $X \bullet_N Y = \gamma_N(X \bullet Y)$, for all operations $\bullet \in \{\circ, \cup, \curlywedge\}$.

Lemma 1. *The structure* $\mathbf{W}_{\mathbf{A},\mathbf{B}} = (W, W', N, \circ, \curlywedge, \varepsilon)$ *supports a distributive residuated frame structure in the sense of [3]. Therefore,*

1. *The algebra* $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ *is a distributive residuated lattice (and* $\curlywedge_N$ *is simply* $\cap$*).*
2. *The map* $b \mapsto \{(id, b)\}^{\lhd}$ *is a (partial algebra) embedding of* $\mathbf{B}$ *into* $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$*.*
3. $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ *is in* $\mathcal{V}$*.*
4. *Every set in* $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ *is an intersection of sets of the form* $\{(u, b)\}^{\lhd}$ *for* $u \in S_W, b \in B$*.*

We will take $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ to play the role of $\mathbf{D}$ in the definition of the FEP. The above lemma provides the embedding and also the membership in $\mathcal{V}$, so the only thing that remains to be shown is the finiteness of $\mathbf{D} = \mathbf{W}^+_{\mathbf{A},\mathbf{B}}$, and this is the content of the next section.

3 Toward finiteness

By Lemma 1(4), it suffices to show that there are only finitely many closed sets of the form $\{(u, b)\}^{\lhd}$ for $u \in S_W, b \in B$. In particular, since B is finite, it suffices to show that for each $b \in B$, the set $C_b = \{\{(u, b)\}^{\lhd} : u \in S_W\}$ is finite. We will show that in C_b, ordered under (reverse) inclusion, all antichains, descending chains and ascending chains are finite, thus yielding finiteness. Toward this goal, we will construct an auxiliary structure $\mathbf{F}$ and a onto homomorphism h from $\mathbf{F}$ to $\mathbf{W}$. $\mathbf{F}$ will be based on the construction of a free meet-semilattice over a poset.

3.1 The construction $\mathcal{M}$

Given a pomonoid $\mathbf{Q}$, we endow the set $\mathcal{M}(\mathbf{Q})$ of all nonempty finitely generated upsets of $\mathbf{Q}$ with the operations $A \wedge B := A \cup B$ and $A \bullet B := \uparrow\{ab : a \in A, b \in B\}$. , for $A, B \in \mathcal{M}(\mathbf{Q})$.

A *semilattice monoid* is an algebra $\mathbf{A} = (A, \wedge, \cdot, 1)$ such that $(A, \wedge)$ is a semilattice, $(A, \cdot, 1)$ is a monoid and multiplication distributes over meet. As usual we define $x \leq y \iff x \wedge y = x$.

Lemma 2. *If* $\mathbf{Q}$ *is a pomonoid, then* $\mathcal{M}(\mathbf{Q})$ *with the above oprations is a semilattice monoid.*

Recall that a poset is said to be *dually well partially ordered* if it has no infinite antichains and no infinite ascending chains.

Lemma 3. *If the* $\mathbf{Q}$ *pomonoid is dually well-partially ordered, then so is* $\mathcal{M}(\mathbf{Q})$*.*

Lemma 4. *If* $\mathbf{P}$ *and* $\mathbf{Q}$ *are pomonoids and* $f : \mathbf{P} \to \mathbf{Q}$ *is a pomonoid homomorphism (order-preserving monoid homomorphism) then* $\bar{f} : \mathcal{M}(\mathbf{P}) \to \mathcal{M}(\mathbf{Q})$ *is a semilattice monoid homomorphism, where* $\bar{f}(A) = \bigwedge_{a \in A} f(a)$.

3.2 The construction of F and finiteness

In [1], it is shown that given a knotted inequality, an equation of the form (a) and a positive integer k, we can construct a k-generated pomonoid $\mathbf{H}$ that is free *for* (but not in) the class of pomonoids that satisfy the knotted inequality and (a). It is further shown in [1] that $\mathbf{H}$ is dually well partially ordered, if $m > n$ in the knotted inequality. We then define $\mathbf{F}$ as $\mathcal{M}(\mathbf{H})$, so Lemma 3 applies. By the k-freeness of $\mathbf{H}$ and Lemma 4, we obtain the following.

Lemma 5. *There is a surjective semilattice-monoid homomorphism* $h : \mathbf{F} \to \mathbf{W}$.

Lemma 6. *For each* $b \in B, (C_b, \supseteq)$ *is a dually well partially ordered set and has no infinite descending chains. Recall* $C_b = \{\{(u, b)\}^{\lhd} : u \in S_W\}$.

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