A description of the localic group of the unit circle

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Abstract. We present the frame $\mathfrak{L}(\mathbb{T})$ of the unit circle by generators and relations as a localic analogue of the quotient space $\mathbb{R}/\mathbb{Z}$. In addition, with an eye on a prospective pointfree description of Pontryagin duality, we show how the usual group operations of the frame of reals can be lifted to the new frame $\mathfrak{L}(\mathbb{T})$, endowing it with a canonical localic group structure.

Keywords: Locale, frame, frame of reals, localic group, frame of the unit circle.

One of the main differences between pointfree topology and classical topology is that the category of locales (the pointfree generalized spaces) has an algebraic dual, the category of frames. This fact allows to present locales by generators and relations in a way familiar from traditional algebra: if S is a set of generators, R is a set of relations $u = v$, where u and v are expressions in terms of the frame operations starting from elements and subsets of S, then there exists a frame $\mathsf{Frm}\langle S|R\rangle$ such that for any frame L, the set of frame homomorphisms $\mathsf{Frm}\langle S|R\rangle \to L$ is in a bijective correspondence with functions $f\colon S \to L$ that turn all relations in R into identities in L.

This fact was used by A. Joyal in order to introduce the pointfree counterpart of the real line [3] which was further studied by B. Banaschewski in [1]. The *frame of reals* is defined as the frame $\mathfrak{L}(\mathbb{R})$ generated by all ordered pairs (p, q) of rationals, subject to the relations

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$,

(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$,

(R4) $\bigvee\{(p, q) \mid p, q \in \mathbb{Q}\} = 1$.

Furthermore, this procedure offers us a natural way to introduce several variants as, for instance, the frame of extended reals and the corresponding lattice of extended real functions studied in [2], and the frame of partial reals and the corresponding lattice of continuous partial real functions introduced in [4] (that arise naturally in the construction of the Dedekind completion of lattices of continuous real functions).

The aim of this talk is to give a presentation of the frame of the unit circle $\mathfrak{L}(\mathbb{T})$ as a member of this family of variants. In particular, we will describe this frame by generators and relations motivated by the classical construction of the unit circle as the quotient space $\mathbb{R}/\mathbb{Z}$.

Besides, with an eye on a prospective pointfree description of the Pontryagin duality, we will show how the localic group structure of the unit circle is obtainable in the pointfree context, by demonstrating how the usual algebraic operations of the frame of reals can be lifted to the new frame.

References

1. B. Banaschewski: The real numbers in pointfree topology. Textos de Matemática, vol. 12, Departamento de Matemática da Universidade de Coimbra (1997).
2. B. Banaschewski, J. Gutiérrez García, J. Picado: Extended real functions in pointfree topology, J. Pure Appl. Algebra 216, 905–922 (2012).
3. A. Joyal: Nouveaux fondaments de l'analyse. Lectures Montréal 1973 and 1974 (unpublished).
4. I. Mozo Carollo, J. Gutiérrez García and J. Picado: On the Dedekind completion of function rings. Forum Math. (doi: 10.1515/forum-2012-0095, in press).
5. J. Picado and A. Pultr: Frames and locales: Topology without points. Frontiers in Mathematics, vol. 28, Springer, Basel (2012).