Coherence for Categories of Posets with Applications

M. Andrew Moshier

Chapman University

Carboni and Walters [1] initiated the study of *cartesian bicategories*. One motivating example in that paper is the category we refer to as Pos^*, consisting of partially ordered sets and *weakening relations*, namely, those relations $R \subseteq A \times B$ between posets for which $a \leq_A a' \; R \; b' \leq_B b$ implies that $a \; R \; b$. Identity for a poset A in Pos^* is the order on A. Composition is relational composition.

Categories that are concrete over Pos^* are of interest for algebraic logic, as one can interpret the order relation on an object as "entailment" and a morphism between objects as a sort of paraphrasis between logics. Dozen and Petric, in [3], consider coherence theorems for proof theory. In that work the categories of interest are logics. In contrast to that, we consider categories in which the objects are the logics and the morphisms are entailment relations *between* logics. Whence our interest is in bicategorical structure of categories of logics, whereas Dozen and Petric are interesting in the (1-)categorical structure of individual logics. The more recent sequel paper of Carboni, Kelley, Walters and Wood [2] generalizes to arbitrary bicategories, and may be useful for combining the ideas of Dozen and Petric with the present work.

The category DL^*, consisting of bounded distributive lattices and relations $R \subseteq A \times B$ that are both weakening relations and sub-bounded lattices of $A \times B$, is another example if a cartesian bicategory (not cited in [1]). These relations can be characterized by a form of Gentzen's sequent rules for positive logic: $a \; R \; b$ implies $a' \wedge a \; R \; b$ [left $\wedge$ rule], $a \; R \; b$ and $R \; b'$ implies $a \; R \; b \wedge b'$ [right $\wedge$ rule], and so on. Composition is a form of cut: $a \; R \; b$ and $b \; S \; c$ implies $a \; R; S \; c$. And on an object, the general form of cut ensures distributivity: $a \leq b \vee c$ and $c \wedge a \leq b$ implies $a \leq b$.

The full subcategory BL^*, consisting of Boolean lattices (complemented distributive lattices) has the added constraint on objects that they obey the negation rule: $a \leq b \vee c$ if and only if $\neg c \wedge a \leq b$. One can independently add other structure such as modal operators. So for example, $\mathsf{DL}^*_{\Diamond\Box}$ ($\mathsf{BL}^*_{\Diamond\Box}$) consists of (complemented) distributive lattices equipped with monotonic operations $\Diamond$ and $\Box$ that are adjoint to each other. The morphisms are relations that are morphisms of DL^* and are closed under $\Box$ and $\Diamond$.

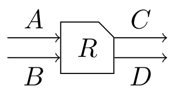
These, and other examples that arise from algebraic logic, are cartesian bicategories. We regard cartesian bicategories as an apt categorial generalization of algebraic propositional logic (at least in situations where the structural rule of weakening is in force), and embark on a study of their general structure. First, we need the basic definition.

Any order enriched category has an internal notion of *map* and *comap*. Namely, two morphisms $\hat{f} \colon A \to B$ and $\check{f} \colon B \to A$ constitute a map/comap pair if $1_A \leq f; g$ and $g; f \leq 1_B$. A *cartesian bicategory*, then, is an order enriched symmetric monoidal category in which each object is equipped with a comonoid $\hat{\delta}_A \colon A \to A \otimes A$ and $\hat{\kappa}_A \colon A \to \mathbb{I}$ so that (i) both $\hat{\delta}_A$ and $\hat{\kappa}_A$ are maps, (ii) the comap $\check{\delta}_A$ corresponding to $\hat{\delta}_A$ satisfies $1_A = \hat{\delta}_A; \check{\delta}_A$, and (iii) every morphism is a lax homomorphism for the comonoid, meaning that $R; \hat{\delta}_B \leq \hat{\delta}_A; (R \otimes R)$ and $R; \hat{\kappa}_B \leq \hat{\kappa}_A$ for any $R \colon A \to B$.

String diagrams, e.g., for symmetric monoidal categories, have proven useful in Physics and Theoretical Computer Science, thanks in part to coherence theorems [4] telling us, roughly, that there is a strict initial category consisting of diagrams equivalent under suitable topological invariants. A good survey of similar diagramatic treatments of coherence is P. Selinger [5]. These results establish a sort of logical completeness for these sorts of categories. Strictness here means that the symmetric monoidal transformations are identities. So for example, $A \otimes (B \otimes C)$ is not merely isomorphic to $(A \otimes B) \otimes C$, it is identical.

Our first result extends a similar courtesy to cartesian bijectories by defining a pre-order on diagrams by rewrite rules. For any given signature, the partial order reflection of this pre-order yields a strict cartesian bicategory that it is initial for cartesian bicategories interpeting that signature.

The diagrams for cartesian bicategories denote morphisms. They consist of "wires" $\xrightarrow{A}$ labelled by objects of the category and "boxes" labelled by morphisms. For example, a morphism $R \colon A \otimes B \to C \otimes D$ may be drawn with two incoming wires labelled A and B and to outgoing wires labeled C and D as in the following.



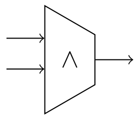
The unit object $\mathbb{I}$ is depicted as an empty diagram, and the tensor of two morphisms is depicted by stacking their diagrams. Composition is depicted by connecting wires while respecting the labels.

The morphisms that characterize cartesian bicategories are depicted by wire splitting and splicing (we omit the object labels here):

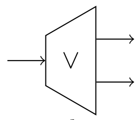
The first coherence theorem states that diagrams of this sort, with a preorder defined by certain rewrite rules, provide the data for an initial cartesian bicategory over a given signature.

The second coherence theorem generalizes this to account for inequational theories. That is, when certain pairs of morphisms (depicted as string diagrams) are interpreted to be in the hom set order relation, the construction yields an initial cartesian bicategory among those that satisfy the given inequations.

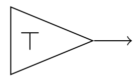
The second coherence theorem is useful for axiomatizing categories that behave like DL^* and BL^*. In particular, say that an object A of a cartesian bicategory is *like a meet semilattice* if $\hat{\delta}_A$ is a comap, and is *like a poset with top* if $\hat{\kappa}_A$ is a comap. Likewise, say that A is *like a join semi-lattice* and *like a poset with bottom* if $\check{\delta}_A$ and $\check{\kappa}_A$ are maps, respectively. In those cases, we use the following symbols to denote the adjoints:



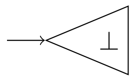
Map adjoint to $\hat{\delta}$



Comap adjoint to $\check{\delta}$



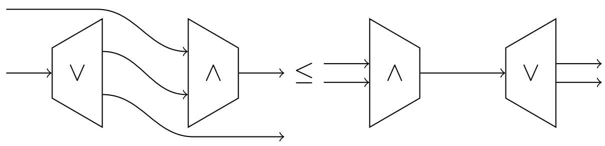
Map adjoint to $\hat{\kappa}$



Comap adjoint to $\check{\kappa}$

It is a useful exercise to prove that if the absorption law holds when whenever $\hat{\delta}_A$ is a comap and $\check{\delta}_A$ is a map.

As an application, we show that in any cartesian bicategory, a lattice-like object obeys the distributive law if and only if



Furthermore, a bounded distributive lattice-like object is complemented if and only if this law is strengthened to be an equivalence of the two diagrams.

In another application, we show that in any cartesian bicategory, a complemented distributive lattice-like object equipped with a single morphism that is both a map and a comap interprets normal modal logic.

References

1. A. Carboni and R.F.C. Walters. "Cartesian Bicategories I". *Journal of Pure and Applied Algebra*, vol. 49 (1987) pp 11-32
2. A. Carboni, G.M. Kelly, R.F.C. Walters, and R.J. Wood. "Cartesian Bicategories II". *Theory and Applications of Categories*, vol. 19 (2008), no. 6, pp 93-124.
3. K. Dozen and Z. Petric *Proof-Theoretical Coherence* College Publications (2004) Later edition available at `www.mi.sanu.ac.rs/~kosta/coh.pdf`
4. G.M. Kelly and M.L. Laplaza. *Coherence for compact closed categories.* Journal Pure Appl. Algebra vol. 19 (1980) pp 193-213.
5. P. Selinger. *A Survey of Graphical Languages for Monoidal Categories* In *New Structures for Physics*, Lecture Notes in Physics, vol. 813 (2011), pp 289-355