Canonicity results for mu-calculi (Unified Correspondence III)

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The modal mu-calculus. The modal mu-calculus was defined in 1983 by Kozen [8] and is obtained by adding the least and greatest fixed point operators to the basic modal logic. An overview of the modal mu-calculus can be found in the chapter by Bradfield and Stirling [4]. Canonical models and completeness results for the finitary and infinitary logics defined by Kozen were obtained by Ambler et al. [1], who introduce two types of algebraic semantics for the mu-calculus: mu-algebras of the first- and second kind. These are special types of Boolean algebras with operators. Specifically, mu-algebras of the first kind require the existence of all meets needed to calculate least fixed points of term functions. Mu-algebras of the second kind additionally require the existence of all joins needed to calculate least fixed points of term functions as ordinal unfoldings.

Preservation under canonical extensions of mu-algebras. We investigate the canonicity of inequalities of the intuitionistic mu-calculus. The language of this logic, denoted by $\mathcal{L}_1$, extends that of intuitionistic modal logic with the least and greatest fixed point binders μ and ν. The preservation under completions of algebras (e.g. the canonical extenstion and the Dedekind-MacNeille completion) of (in)equalities containing least fixed points has been the subject of some recent research, including [3] and [2]. Proving such preservation results poses some interesting challenges, as the standard techniques tend to founder on the fact that, unlike the operations interpreting the algebraic signature, fixed points in the completions are *not* extensions of those in the original algebra.

One way to circumvent this difficulty is to adjust the definition of canonicity, by specifying that fixed points in the canonical extension are to be interpreted as the meets of pre-fixed points *of the small algebra*. In this way fixed points *do* extend from a mu-algebra to its canonical extension. This strategy is employed in [3], there formulated in terms of general frames. We will refer to this notion of canonicity as *tame canonicity*.

Syntactic classes. The work we report on here is due to appear as [5]. Our approach is in the spirit of Sahlqvist theory. That is, we identify syntactically-defined classes of inequalities, namely the *restricted inductive* and *tame inductive inequalities*, which we show to be, respectively, canonical or tame canonical. Both the restricted inductive and tame inductive inequalities are subclasses of the *inductive inequalities of the intuitionistic mu-calculus* as defined in [6], where it is proved that all such inequalities have local frame correspondents in the language of first-order logic with least fixed points. We compare our classes with others to be found in the literature. Particularly, we show that the tame inductive inequalities, when projected onto the classical case, strictly extended the class given in [3], with formulas like $(\Diamond(\Box\bot \vee p) \wedge \Box q) \rightarrow (\mu Y.(\Diamond(p \wedge q) \wedge \Box Y))$.

The calculus and algorithm μ^*-ALBA. We introduce an algorithm, called μ^*-ALBA, which processes inequalities with the aim of eliminating their propositional variables, by transforming them into *pure inequalities* in an extended *hybrid language.* An inequality for which this succeeds is called a *μ^*-ALBA inequality.*

μ^*-ALBA is closely related to, but different from, the algorithms ALBA and mu-ALBA studied by Conradie, Palmigiano, et al in [7] and [6]. It is based on a calculus of rewrite rules, the soundness of which rests upon the way in which mu-algebras of the first- and second kind embed into their canonical extensions and the order-theoretic properties of the latter. This requires some novel results on the order-theoretic behaviour in these algebras of term functions involving fixed point binders.

By imposing certain restrictions on the application of rewrite rules we obtain the notions of *tame* and *proper runs* of μ^*-ALBA.

Canonicty of μ^*-ALBA-inequalities. By applying the order theoretic and order-topological results mentioned in the paragraph above we obtain the following canonicity theorems for μ^*-ALBA inequalities:

Theorem 1. (Tame Canonicity) *All $\mathcal{L}_1$-inequalities on which a tame run of μ^*-ALBA succeeds are tame canonical.*

To obtain canonicity results in the usual sense, one needs to restrict to mu-algebras of the second kind:

Theorem 2. (Canonicity) *Let $\mathbb{A}$ be a mu-algebra of the second kind and let $\varphi \leq \psi$ be an $\mathcal{L}_1$-inequality on which a proper run of μ^*-ALBA succeeds. If $\mathbb{A} \models \varphi \leq \psi$ then $\mathbb{A}^\delta \models \varphi \leq \psi$.*

Canonicty of restricted- and tame inductive inequalities. First we show that μ^*-ALBA succeeds on every restricted inductive inequality by means of a proper run, and on every tame inductive inequality by means of a tame run. Combining these result with Theorems 1 and 2 above, we obtain our main theorem as a corollary:

Theorem 3. *All restricted inductive mu-inequalities are canonical and all tame inductive mu-inequalities are tame canonical.*

References

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