A uniform continuity principle for the Baire space and a corresponding bar induction

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The uniform continuity principle (UC) is the following statement:

UC Every pointwise continuous function $F: \{0,1\}^\mathbb{N} \rightarrow \mathbb{N}$ is uniformly continuous.

Since the Cantor space $\{0,1\}^\mathbb{N}$ is compact, UC is classically true. It is well known, however, that this need not be the case in the constructive mathematics in the sense of Bishop [2].

Berger [1] showed that, in Bishop constructive mathematics, UC is equivalent to a version of fan theorem, called $c$–FT, thereby gave a characterisation of UC in terms of a fan theorem. In this talk, we generalise the above equivalence to the setting of the Baire space $\mathbb{N}^\mathbb{N}$. We formulate the corresponding uniform continuity principle $UC_B$ for the Baire space and a version of bar induction, called $c$–BI, and show that $UC_B$ and $c$–BI are equivalent.

The principle $UC_B$ is the following statement:

$UC_B$ Every pointwise continuous function $F: \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}$ is formally representable.

Here, a function $F: \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}$ is formally representable if $F$ is of the form $\mathcal{P}(r): \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}$ for some morphism $r : \mathcal{B} \rightarrow \mathcal{N}$ from the formal Baire space $\mathcal{B}$ to the formal discrete space $\mathcal{N}$ of natural numbers in the category of formal topologies $FTop$ [4] (or that of formal spaces [3]), where $\mathcal{P}: FTop \rightarrow Top$ is the right adjoint of the standard adjunction between the category of topological spaces $Top$ and $FTop$.

The principle $c$–BI reads as follows:

$c$–BI For any $c$–bar $P \subseteq \mathbb{N}^*$ and a predicate $Q \subseteq \mathbb{N}^*$, if $P \subseteq Q$ and $Q$ is inductive, then $Q(\langle \rangle)$.

Here, a predicate $P \subseteq \mathbb{N}^*$ on the finite sequences of $\mathbb{N}$ is a $c$–bar if

1. $P$ is a bar, i.e. $(\forall \alpha \in \mathbb{N}^*) (\exists n \in \mathbb{N}) P(\langle \alpha(0), \ldots, \alpha(n-1) \rangle),$
2. there exists a function $\delta: \mathbb{N}^* \rightarrow \mathbb{N}$ such that

$$(\forall a \in \mathbb{N}^*) [P(a) \iff (\forall b \in \mathbb{N}^*) \delta(a) = \delta(a * b)],$$

and a predicate $Q \subseteq \mathbb{N}^*$ is inductive if $(\forall a \in \mathbb{N}^*) [(\forall n \in \mathbb{N}) Q(a * \langle n \rangle) \rightarrow Q(a)].$

We have $BI_M \Rightarrow c$–BI $\Rightarrow BI_D$, where $BI_M$ is the monotone bar induction and $BI_D$ is the bar induction for decidable bars [5].

The equivalence between $UC_B$ and $c$–BI can be seen as a generalisation of that of Berger in the light of the following observations.
1. A function $F : \{0,1\}^N \to \mathbb{N}$ is uniformly continuous if and only if $F$ is formally representable by some morphism $r : C \to \mathcal{N}$ from the formal Cantor space $C$ to $\mathcal{N}$.

2. If we replace $\mathbb{N}$ with $\{0,1\}$ everywhere in the statement of $c$–BI, we obtain a version of fan theorem which is equivalent to $c$–FT.

We work in Bishop constructive mathematics with the axiom of countable choice and generalised inductive definitions which have rules with countable premises.

References


