A uniform continuity principle for the Baire space and a corresponding bar induction

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The uniform continuity principle (**UC**) is the following statement:

UC Every pointwise continuous function $F\colon \{0,1\}^{\mathbb{N}} \to \mathbb{N}$ is uniformly continuous.

Since the Cantor space $\{0,1\}^{\mathbb{N}}$ is compact, **UC** is classically true. It is well known, however, that this need not be the case in the constructive mathematics in the sense of Bishop [2].

Berger [1] showed that, in Bishop constructive mathematics, **UC** is equivalent to a version of fan theorem, called **c–FT**, thereby gave a characterisation of **UC** in terms of a fan theorem. In this talk, we generalise the above equivalence to the setting of the Baire space $\mathbb{N}^{\mathbb{N}}$. We formulate the corresponding uniform continuity principle $\mathbf{UC_B}$ for the Baire space and a version of bar induction, called **c–BI**, and show that $\mathbf{UC_B}$ and **c–BI** are equivalent.

The principle $\mathbf{UC_B}$ is the following statement:

$\mathbf{UC_B}$ Every pointwise continuous function $F\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is formally representable.

Here, a function $F\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is *formally representable* if F is of the form $\mathcal{P}t(r)\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ for some morphism $r : \mathcal{B} \to \mathcal{N}$ from the formal Baire space $\mathcal{B}$ to the formal discrete space $\mathcal{N}$ of natural numbers in the category of formal topologies **FTop** [4] (or that of formal spaces [3]), where $\mathcal{P}t : \mathbf{FTop} \to \mathbf{Top}$ is the right adjoint of the standard adjunction between the category of topological spaces **Top** and **FTop**.

The principle **c–BI** reads as follows:

c–BI For any c–bar $P \subseteq \mathbb{N}^*$ and a predicate $Q \subseteq \mathbb{N}^*$, if $P \subseteq Q$ and Q is inductive, then $Q(\langle\rangle)$.

Here, a predicate $P \subseteq \mathbb{N}^*$ on the finite sequences of $\mathbb{N}$ is a *c–bar* if

1. P is a bar, i.e. $\left(\forall \alpha \in \mathbb{N}^{\mathbb{N}}\right)(\exists n \in \mathbb{N})\, P(\langle \alpha(0), \ldots, \alpha(n-1)\rangle)$,
2. there exists a function $\delta\colon \mathbb{N}^* \to \mathbb{N}$ such that

$$(\forall a \in \mathbb{N}^*)\left[P(a) \leftrightarrow (\forall b \in \mathbb{N}^*)\, \delta(a) = \delta(a * b)\right],$$

and a predicate $Q \subseteq \mathbb{N}^*$ is *inductive* if $(\forall a \in \mathbb{N}^*)\left[(\forall n \in \mathbb{N})\, Q(a * \langle n\rangle) \to Q(a)\right]$. We have $\mathbf{BI_M} \Rightarrow$ **c–BI** $\Rightarrow \mathbf{BI_D}$, where $\mathbf{BI_M}$ is the monotone bar induction and $\mathbf{BI_D}$ is the bar induction for decidable bars [5].

The equivalence between $\mathbf{UC_B}$ and **c–BI** can be seen as a generalisation of that of Berger in the light of the following observations.

1. A function $F\colon \{0,1\}^{\mathbb{N}} \to \mathbb{N}$ is uniformly continuous if and only if F is formally representable by some morphism $r\colon \mathcal{C} \to \mathcal{N}$ from the formal Cantor space $\mathcal{C}$ to $\mathcal{N}$.
2. If we replace $\mathbb{N}^{\mathbb{N}}$ with $\{0,1\}^{\mathbb{N}}$ and $\mathbb{N}^*$ with $\{0,1\}^*$ everywhere in the statement of **c–BI**, we obtain a version of fan theorem which is equivalent to **c–FT**.

We work in Bishop constructive mathematics with the axiom of countable choice and generalised inductive definitions which have rules with countable premises.

References

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