Bisimulation games and locally tabular modal logics

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In this talk we present new results on local tabularity for modal logics (or equivalently, local finiteness of varieties of modal algebras); for their proofs we use bisimulation games.

We consider an N-modal propositional language, with formulas built from the countable set of proposition letters $\{p_1, p_2, \dots\}$ and the connectives $\to, \bot, \Box_1, \dots, \Box_N$. Other connectives ($\wedge,\ \vee,\ \neg,\ \top,\ \leftrightarrow, \Diamond_i$) are defined as standard abbreviations. We also use the 'joined' box and diamond:

$$\Box A := \Box_1 A \wedge \ldots \wedge \Box_N A,\ \ \Diamond A := \Diamond_1 A \vee \ldots \vee \Diamond_N A.$$

As usual, $\Box^n$ means $\underbrace{\Box \ldots \Box}_{n}$.

A *k-formula* is a formula using only the proposition letters from the set $p_1, p_2, \dots p_k$.

The *modal depth* $md(A)$ of a modal formula A is defined by induction:

$$md(\bot) = md(p_i) = 0,\ md(A \to B) = \max(md(A), md(B)),$$
$$md(\Box_j A) = md(A) + 1.$$

The definitions of (normal) modal logics, Kripke frames and validity are standard. $\mathbf{L}(\mathcal{C})$ denotes the modal logic determined by a class of frames $\mathcal{C}$ (i.e., the set of all modal formulas valid in $\mathcal{C}$). $\mathbf{K}_N$ denotes the minimal N-modal logic; $\mathbf{K} = \mathbf{K}_1$.

The restriction of a modal logic L to k-formulas is denoted by $L\lceil k$; the sets $L\lceil k$ are called *k-weak modal logics*. Respectively, in *k-weak Kripke models* only the letters $p_1, p_2, \dots p_k$ (and k-formulas) are evaluated.

Definition 1. *For a frame $F = (W, R_1, \dots, R_N)$ the relation $R_1 \cup \ldots \cup R_N$ is denoted by R. A* path of length m *from u to v in F is a sequence of points $(u_0, u_1, \dots, u_m)$, in which $u = u_0,\ v = u_m$ and $u_i R u_{i+1}$ for all $i < m$; a singleton sequence (u) is a path of length 0.*

A path is called simple *if all its points are different.*

A simple chain *in a transitive frame (W, R) is a path $(u_0, u_1, \dots, u_m)$, in which $u_{i+1} \not R u_i$ for all $i < m$.*

Definition 2. *The* depth $d(F)$ *of a frame F is the maximum of lengths of paths in F (if it exists), or ∞ otherwise. The* simple depth $\underline{d}(F)$ *of F is the maximum of lengths of simple paths in F (if it exists). For a transitive frame F the* transitive depth $td(F)$ *is the maximal length of simple chains (if it exists).*

Recall a syntactic description of these notions. Put

$$P_i^n := p_i \wedge \bigwedge\{\neg p_j \mid j \neq i,\ 0 \leq j \leq n\} \text{ for } 0 \leq i \leq n;$$
$$C\alpha_{n,N} := \neg(P_0^n \wedge \Diamond(P_1^n \ldots \wedge \Diamond P_n^n) \ldots));$$
$$bd_1 := \Diamond\Box p_1 \to p_1,\ bd_{n+1} := \Diamond(\Box p_{n+1} \wedge \neg bd_n) \to p_{n+1}.$$

Lemma 1 *(1)* $d(F) < n$ iff $F \vDash \Box^n \bot$.
(2) $\underline{d}(F) < n$ *iff* $F \vDash C\alpha_{n,N}$.
(3) $td(F) < n$ *iff* $F \vDash bd_n$.

Formulas $C\alpha_{n,N}$ are polymodal versions of formulas α_n used in Chagrov's tabularity criterion from [2].

Definition 3. *A modal logic determined by a single finite frame is called* tabular.

A modal logic has the finite model property (fmp) *if it is an intersection of tabular logics.*

An N-modal logic L is called locally tabular *if for any finite k there exist finitely many N-modal k-formulas up to equivalence in L.*

In algebraic terms, tabularity of L of means that the corresponding variety of L-algebras is generated by a single finite algebra. Local tabularity means the *local finiteness* of the variety of L-algebras, i.e., finiteness of all finitely generated L-algebras, cf. [6].

Recall some well-known facts:

Proposition 2 *(1) L is locally tabular iff every weak canonical model $M_{L\lceil k}$ is finite.*
(2) Tabularity and local local tabularity are inherited by extensions in the same language.
(3) Tabularity implies local tabularity, and local tabularity implies the fmp.

Theorem 3. *(Cf. [1]) For a weak Kripke model M the following conditions are equivalent:*

(1) $M, x \vDash A$ iff $M, x' \vDash A$ for any formula A of modal depth $\leq n$;
(2) the Duplicator has a winning strategy in the bisimulation game of length n in M with the initial position (x, x').

The equivalence relation from this theorem (*n-bisimilarity*) is denoted by $M, x \equiv_n M, x'$, or by $x \equiv_n x'$ if M is clear from the context.

Proposition 4 *In every weak Kripke k-model the number of n-bisimilarity classes is finite; it is bounded by a function depending only on n and k.*

Definition 5 *The* modal depth $md(L)$ *of a modal logic L is the minimal n such that in L every formula is equivalent to a formula of modal depth $\leq n$ (or ∞ if such n does not exist).*

Then we readily have

Proposition 6 *If $md(L) < \infty$, then L is locally tabular.*

To estimate $md(L)$ one can use bisimulation games, thanks to the following observation:

Proposition 7 $md(L) \leq n$ *iff* $\equiv_n = \equiv_{n+1}$ *in every weak canonical model of* L.

Theorem 8. *Every tabular modal logic is of finite modal depth: if F is a finite frame of cardinality n, then $md(\mathbf{L}(F)) \leq n^2 + 1$.*

The next theorem mentions the difference logic **DL** (whose frames are sets with the inequality relations) and **Grz3**, the logic determined by finite linear orders.

Theorem 9. *(1)* $md(\mathbf{K}_N + \Box^n \bot) = n - 1$.
(2) $md(\mathbf{DL}) = 2$.
(3) $md(\mathbf{K4} + bd_n) \leq 4n - 3$.
(4) $md(\mathbf{Grz3} + bd_n) \leq n - 1$.

Note that (3) implies $md(\mathbf{S5}) = 1$, which is well-known. (3) also implies the local tabularity of $\mathbf{K4} + bd_n$ (Segerberg's theorem, cf. [4]).

Definition 10 *(cf.[3]) The* commutative join $[L_1, L_2]$ *of an N_1-modal logic L_1 and an N_2-modal logic L_2 is obtained from their fusion by adding the axioms*

$$\Diamond_i \Box_{r+j} p \to \Box_{r+j} \Diamond_i p, \;\; \Box_i \Box_{r+j} p \leftrightarrow \Box_{r+j} \Box_i p$$

for $1 \leq i \leq N_1$, $1 \leq j \leq N_2$.

Recall that the corresponding frame conditions are:

$$R_i^{-1} \circ R_{r+j} \subseteq R_{r+j} \circ R_i^{-1}, \;\; R_{r+j} \circ R_i = R_i \circ R_{r+j}.$$

Theorem 11. *If $md(L) = m$, then $md([\mathbf{K}_N + \Box^n \bot, L]) \leq (m+1)n - 1$.*

Theorem 12. *Every logic $\mathbf{K}_N + C\alpha_{n,N}$ is locally tabular and moreover, the logics $[\mathbf{K}_N + C\alpha_{n,N}, \mathbf{K}_{N_1} + \Box^n \bot]$, $[\mathbf{K}_N + C\alpha_{n,N}, \mathbf{S5}]$ are locally tabular.*

This theorem in particular implies the local tabularity of the temporal **K4**-extensions of $\mathbf{K}_2 + C\alpha_{n,2}$ stated in [2].

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