MV-pairs and state operators

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Abstract. Flaminio and Montagna (2008) enlarged the language of MV-algebras by a unary operation σ, called internal state or a state operator, equationally defined so as to preserve the basic properties of a state in its usual meaning. The resulting class of MV-algebras is called state MV-algebras. Jenča (2007) and Vetterlein (2008), using different approaches, represented MV-algebras through the quotient of a Boolean algebra B by a suitable subgroup G of the group of all automorphisms of B. Such a couple (B, G) is called an MV-pair. We introduce the notion of a state MV-pair as a triple (B, G, σ), where (B, G) is an MV-pair and σ is a state operator on B, and show that there are relations between state MV-pairs and state MV-algebras similar to the relations between MV-pairs and MV-algebras. We also give a characterization of those MV-pairs, resp. state MV-pairs, that induce subdirectly irreducible MV-algebras, resp. state MV-algebras.

MV-algebras were introduced as algebraic bases for many-valued logic [3]. That is, MV-algebras stand in relation to the Łukasziewicz infinite valued logic as Boolean algebras stand to classical two-valued logic. A key relationship between Boolean algebras and MV-algebras lies in the fact that the set of all idempotents of an MV-algebra M is a Boolean algebra, in fact the greatest Boolean subalgebra of M. The Boolean algebra of idempotents can be considered as a system of classical propositions, while the surrounding algebra M can be considered as an extension of the classical logic by fuzzy, resp. unsharp propositions.

Another relation between MV-algebras and Boolean algebras was described in [10], where a representation theorem for MV-algebras is given in terms of Boolean algebras and their automorphism groups. Actually, it is shown in [10] that given a Boolean algebra B and a subgroup G of its automorphism group satisfying certain conditions, the pair (B, G) can be canonically associated with an MV-algebra. Such pairs (B, G) are called MV-pairs. Conversely, given an MV-algebra M, if $B(M)$ denotes its R-generated Boolean algebra [11] and $G(M)$ is a special subgroup of the automorphism group of $B(M)$, it turns out that $(B(M), G(M))$ forms an MV-pair. In [7], a categorical development of the results in [10] is presented.

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MV-algebras are equivalent to MV-effect algebras [4], which are a special subclass of a more general class of effect algebras [9]. Equivalence between MV-algebras and MV-effect algebras was used in [10], where it was proved that given an MV-pair (B, G), G induces a congruence on B such that the quotient B/G is an MV-effect algebra.

States on MV-algebras, as averaging of truth values, were studied in [12]. Recently, the notion of a state was generalized in [8] to an algebraically defined notion for MV-algebras. The language of MV-algebras has been enlarged by a unary operation σ, called an internal state or a state operator. Such MV-algebras are called state-MV-algebras. These algebras are now intensively studied, e.g. [6]. In [2], a generalization of the notion of a state operator for effect algebras was introduced and studied. The authors also introduced the notion of a strong state operator, and have shown that the effect algebra state operator coincides, on MV-effect algebras, with the MV-algebra state operator as introduced in [8] if and only if it is strong.

The present contribution is based on [1], where internal states in connection with MV-pairs are studied. Namely, we study relations between state MV-algebras and state Boolean algebras, which are connected by an MV-pair. One of the main results is the following [1, Theorem 4.11]:

Theorem 1. *If (B, G, σ) is a (strong) state-MV-pair, then $M = B/G$ is a (strong) state-MV-effect algebra with the state operator $\sigma_*([a]_G) = [\sigma(a)]_G$. Conversely, if (M, σ) is a (strong) state-MV-effect algebra, then $(B(M), G(M), \sigma^*)$, where $\sigma^*(a) = \sigma([a]_{G(M)})$, is a strong state-MV-pair and $(\sigma^*)_* = \sigma$.*

The outline of the talk is as follows.

1. We introduce some definitions and known results that we need in what follows.

2. We introduce a new definition of morphisms of MV-pairs (which, in our opinion is more transparent), and we show that it is equivalent with the original definition introduced in [7].

3. We introduce the notions of state MV-pairs and strong state MV-pairs, and study their relations with state MV-algebras.

4. We find conditions under which an MV-pair, resp. a state MV-pair, gives rise to a linearly ordered and to a subdirectly irreducible MV-algebra, resp. a subdirectly irreducible state-MV-algebra.

References

1. S. Pulmannová, E. Vinceková: *MV-pair and state operator*, Fuzzy Sets and Sysrtems,**260** (2015), 62-76.
2. D. Buhagiar, E. Chetcuti, A. Dvurečenskij: *Loomis-Sikorski theorem and Stone duality for effect algebras with internal state*, Fuzzy Sets and Systems **172** (2011), 71–86.
3. C.C. Chang: *Algebraic analysis of many-valued logics*, Trans. amer. Math. Soc. **88** (1958), 467–490.

4. F. Chovanec, F. Kôpka: *Boolean D-posets*, Tatra Mountains Math. Publ. **10** (1997), 1–15.
5. R.L. Cignoli, I.M. D'Ottaviano, D. Mundici: *Algebraic Foundations of Many-Valued Reasoning.* Springer, 2000 edition, 1999.
6. A. Di Nola, A. Dvurečenskij: *State-morphism MV-algebras*, J. Pure Appl. Logic **161** (2009), 161–173.
7. A. Di Nola, M. Holčapek, G. Jenča: *The category of MV-pairs*, Logic Journal of the IGLP **17** (2009), 395–412.
8. T. Flaminio, F. Montagna: *MV-algebras with internal states and probabilistic fuzzy logics*, International Journal of Approximate Reasoning **50** (2009), 138–152.
9. D.J. Foulis, M.K. Bennett: *Effect algebras and unsharp quantum logics*, Found. Phys. **24** (1994), 1325–1346.
10. G. Jenča: *A representation theorem for MV-algebras*, Soft Computing **11** (2007), 557–564.
11. G. Jenča: *Boolean algebras R-generated by MV-effect algebras*, Fuzzy sets and systems **145** (2004), 279–285.
12. D. Mundici: *Averaging the truth-value in Łukasziewicz logic*, Studia Logica **55** (1995), 113–127.
13. T. Vetterlein: *Boolean algebras with an automorphism group: a framework for Łukasziewicz logic*, J. Multi-Valued log. Soft Comput. **14** (2008), 51-67.