On Σ_2^0 Relations and Elementary Embeddability at Uncountable Cardinals

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We prove a dichotomy property related to variants of the model theoretic spectrum functions by studying topological properties of the generalized Baire spaces ${}^{\kappa}\kappa$ and using methods from algebraic logic. For a given first order theory T, the spectrum function correlates to each κ the cardinality of the set of κ-sized models of T up to isomorphism. Our variants are obtained by replacing the role of isomorphism with that of embeddability, elementary embeddability or similar concepts and restricting the set of "allowed" embeddings. In the case of countable theories, the set of possible spectrum functions has been completely characterized in [1] and the variant with elementary embeddability has also been studied in [4]. Our results and methods were motivated by those found in [3], where the case of countable models and the corresponding variant of Vaught's conjecture is studied.

From now on, let κ be a fixed uncountable cardinal and L a first order relational language of size at most κ. We investigate the set $\mathrm{Mod}_{\kappa}^{\psi}$ of models of an $L_{\kappa^+\kappa}$-sentence ψ with domain κ, when considered up to isomorphism, embeddability, and elementary embeddability by elements of a certain submonoid H of ${}^{\kappa}\kappa$ (the set of functions $\kappa \to \kappa$). More generally, ψ can be a sentence in $\mathbf{\Sigma}_1^1(L_{\kappa^+\kappa})$, and instead of (elementary) embeddings, we can consider functions in H preserving all formulas in a *fragment* F of $L_{\kappa^+\kappa}$; such functions will be called (F,H)*-elementary embeddings.* By a *fragment*, we mean a κ-sized set of $L_{\kappa^+\kappa}$-formulas which contains all the atomic formulas and is closed under negation and finite conjunction, taking subformulas, and substitution of variables. Notice that both embeddability and elementary embeddability by H are special cases of (F,H)-elementary embeddability, and when H is a subgroup of $Sym(\kappa)$ this concept reduces to isomorphism by H. Further special cases that could be of interest are obtained when $F = L_{\lambda\mu}$, where $\omega \leq \mu \leq \lambda \leq \kappa$ or when F is the n-variable logic $L_{\lambda\mu}^n$, (where $n < \omega$).

We obtain a dichotomy theorem for models up to (F,H)-elementary embeddability as a special case of a more general dichotomy for certain binary relations of the generalized Baire space, which we prove holds under an additional set theoretical assumption (see Theorem 1 and Corollary 2 below). The domain of the *generalized Baire space associated to* κ is ${}^{\kappa}\kappa$ and its topology is given by the basic open sets $N_p = \{x \in {}^{\kappa}\kappa : p \subseteq x\}$ for all $p \in {}^{<\kappa}\kappa$. Our model theoretic dichotomy says that if $H \subseteq {}^{\kappa}\kappa$ is "nice" enough and there are κ^+-many pairwise non (F, H)-elementarily embeddable models in $\mathrm{Mod}^{\psi}_{\kappa}$, then there are *$\kappa$-perfectly many* such models. The conclusion of the previous statement means that, in some topological sense, there are as many "different" models as possible, and is a natural generalization to the uncountable case of the concept of perfectly many models. In particular, it implies the existence of 2^{κ} many "different" models.

With a modification of an argument in [3] from the representation theory of locally finite cylindric algebras, we can embed the set $\mathrm{Mod}^{\psi}_{\kappa}$ into ${}^{\kappa}\kappa$. In more detail, there is a one-one correspondence between the set Mod_{κ} of all models for the language L with domain κ and a certain subset of the Stone dual of the Lindenbaum-Tarski algebra associated to all formulas in the fragment F. This subset is defined by conditions on the ultrafilters which correspond to the inductive definition of the satisfaction of the formulas in F. Therefore the characteristic function of the ultrafilters in the above subset give us an embedding of Mod_{κ} into the subspace ${}^{\kappa}2$ of ${}^{\kappa}\kappa$ (since $|F| = \kappa$). By further properties of fragments, the image of this embedding is a $\mathbf{\Pi}^0_2$ subset of the κ-Baire space ${}^{\kappa}\kappa$ (meaning that it is the intersection of κ many open sets), and therefore $\mathrm{Mod}^{\psi}_{\kappa}$ can be viewed as an analytic subset of the κ-Baire space (i.e., the continuous image of a κ-Borel subset of the κ-Baire space). Furthermore, for "nice" submonoids H of the κ-Baire space, (F, H)-elementary embeddability corresponds to a $\mathbf{\Sigma}^0_2$ binary relation on $\mathrm{Mod}^{\psi}_{\kappa}$, i.e., one that is a union of κ many closed subsets. Here, "nice" means that H is a K_{κ} subset of the κ-Baire space, i.e., the union of at most κ many κ-compact subsets. (A subset is *κ-compact* if any open cover of it has a subcover of size $< \kappa$.)

Thus, the dichotomy we search for follows from a more general one for arbitrary $\mathbf{\Sigma}^0_2$ binary relations on analytic subsets of the κ-Baire space. By our Theorem 1 below, such a dichotomy holds (even for arbitrary n-ary relations) under an additional set theoretic hypothesis $\mathrm{I}^-(\kappa)$ when κ is inaccessible, and under $\mathrm{I}^-(\kappa)$ and the combinatorial principle $\diamondsuit_{\kappa}$ when κ is arbitrary. The hypothesis $\mathrm{I}^-(\kappa)$ is a modification of the hypothesis $\mathrm{I}(\kappa)$ found in literature (see e.g. [2]) and states the following:

> there exists a κ^+-complete normal ideal $\mathcal{I}$ on κ^+ and a dense subset $K \subseteq \mathcal{I}^+$ such that every descending sequence of elements of K of length $< \kappa$ has a lower bound in K.

The assumption that $\mathrm{I}^-(\kappa)$ and $\diamondsuit_{\kappa}$ both hold is consistent relative to the existence of a measurable cardinal $\lambda > \kappa$.

Theorem 1. *Assume $\mathrm{I}^-(\kappa)$ and either that $\diamondsuit_{\kappa}$ or that κ is inaccessible. Suppose $R \subseteq {}^{n}X$ is a $\mathbf{\Sigma}^0_2$ subset, where $1 < n < \omega$ and X is an analytic subset of ${}^{\kappa}\kappa$. Then either all R-independent sets are of size at most κ, or there is a κ-perfect R-independent set.*

Corollary 2. *Assume $\mathrm{I}^-(\kappa)$ and either that $\diamondsuit_{\kappa}$ or that κ is inaccessible. Let H be a K_{κ} submonoid of the κ-Baire space, F a fragment of $L_{\kappa^+\kappa}$ and ψ a sentence of $\mathbf{\Sigma}^1_1(L_{\kappa^+\kappa})$. If there are at least κ^+ many pairwise non (F, H)-elementarily embeddable models in $\mathrm{Mod}^{\psi}_{\kappa}$, then there are κ-perfectly many such models.*

References

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