On two ways of generating topological spaces from Grzegorczyk mereological structures

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1 Grzegorczyk connection structures

The first way was first described in [4] and was later presented and analyzed in [2] and [3]. It is a theory of triples \( (M, \sqsubseteq, \mathcal{C}) \) (with \( \sqsubseteq \) being called part of relation and \( \mathcal{C} \) – connection relation) such that \( (M, \sqsubseteq) \) is a poset with unity which satisfies the following axioms:

\[
\begin{align*}
  x \not\sqsubseteq y \rightarrow \exists z \in M (z \sqsubseteq x \land z \perp y \land \forall u \in M (u \sqsubseteq x \land u \perp y \rightarrow u \sqsubseteq z)), & \quad (\text{SSP}+) \\
  \forall x,y \in M \exists u \in M \sup \{x,y\}, & \quad (\exists \sup) \\
  \forall x,y \in M (x \bigcirc y \rightarrow \exists u \in M \inf \{x,y\}). & \quad (\exists \inf)
\end{align*}
\]

where:

\[
\begin{align*}
  x \sqsubset y & \iff x \sqsubseteq y \land x \neq y, & \quad (\text{df } \sqsubset) \\
  x \bigcirc y & \iff \exists z \in M (z \sqsubseteq x \land z \sqsubseteq y), & \quad (\text{df } \bigcirc) \\
  x \perp y & \iff \neg \exists z \in M (z \sqsubseteq x \land z \sqsubseteq y). & \quad (\text{df } \perp)
\end{align*}
\]

The above relations are called, respectively, proper part, compatibility and incompatibility relation. \( \sup \) and \( \inf \) are the standard supremum and infimum relations in posets. Every triple \( (M, \sqsubseteq, \mathcal{C}) \) satisfying the above set of postulates will be called Grzegorczyk mereological structure.\(^1\)

The connection relation \( \mathcal{C} \) is to satisfy the following three postulates:

\[
\begin{align*}
  \forall x,y \in R (x \sqsubseteq y \rightarrow x \sqsubset y), & \quad (C1) \\
  \forall x,y \in R (x \sqsubset y \rightarrow y \sqsubset x), & \quad (C2) \\
  \forall x,y,z \in R (x \sqsubseteq y \land z \sqsubset x \rightarrow z \sqsubset y). & \quad (C3)
\end{align*}
\]

\(^1\) The system was first presented in [5] in slightly different form. For a full presentation and analysis see [6].

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In \( <M, \sqsubseteq, C> \) we introduce binary relation of non-tangential part:
\[
x \ll y \iff \forall z \in M (z \perp y \rightarrow \neg z \sqsubseteq x).
\] (df \ll)

By means of the auxiliary relation we define a representative of a point\(^2\) as a set \( \emptyset \neq Q \subseteq M \) such that:
\[
\begin{align*}
\forall u,v \in Q (u \neq v & \rightarrow u \ll v \lor v \ll u), \tag{R1} \\
\forall u \in Q \exists v \in X v \ll u, \tag{R2} \\
\forall u,v \in R (\forall z \in Q (z \circ u \land z \circ v) \rightarrow u \sqsubseteq v). \tag{R3}
\end{align*}
\]

The set of representatives of points will be denoted by \( 'Q' \). We now introduce the so called Grzegorczyk axiom:
\[
x \sqsubseteq y \rightarrow \exists Q \subseteq x (x \circ y \rightarrow \exists z \subseteq x \sqsubseteq x \cap y) \land \forall z \subseteq x (z \circ x \land z \circ y), \tag{G}
\]
and call every Grzegorczyk mereological structure satisfying axioms (C1)-(C3), (G) Grzegorczyk connection structure. The class of all such structures will be denoted by \( 'G' \).

In every structure from \( G \) we define points as filters generated by representatives of points:
\[
\Pi := \{ X \subseteq M \mid \exists Q \subseteq x (x \subseteq M \land \exists y \subseteq Q y \subseteq x) \}. \tag{df \Pi}
\]

Elements of \( \Pi \) will be denoted by small Greek letters: \( '\alpha', '\beta', '\gamma', '\delta' \). With every element of the domain of a structure from the class \( G \) we associate the set of its internal points:
\[
\text{Irl}(x) := \{ \alpha \in \Pi \mid x \in \alpha \}. \tag{df Irl}
\]

The set \( \{ \text{Irl}(x) \mid x \in M \} \) is a basis, therefore the pair \( \langle \Pi, \emptyset \rangle \) where:
\[
A \in \emptyset \iff \exists A \subseteq \emptyset A = \bigcup A
\]
is a topological space.

## 2 Roep er connection structures

The second way of constructing points and topological spaces originates from the ideas presented in [7]. This time structures we focus upon are of the form \( <M, L, \sqsubseteq, C> \) where:
- \( <M, \sqsubseteq> \) is a Grzegorczyk mereological structure,
- \( C \) satisfies (C1)-(C3) plus the following:
\[
\forall x,y,z \in R (x \sqsubseteq y \cup z \rightarrow x \sqsubseteq z \lor x \sqsubseteq y), \tag{C4}
\]

\(^2\) Grzegorczyk's representative of points are very similar to Whitehead's more known abstractive sets from [8].
- \( L \) is a subset of \( M \) whose elements are called \textit{limited regions} and satisfy the following postulates:

\[
\begin{align*}
\text{(L1)} & \quad x \in L \land y \subseteq x \implies y \in L, \\
\text{(L2)} & \quad x \in L \lor y \in L \implies x \sqcup y \in L, \\
\text{(L3)} & \quad x \sqsubseteq y \implies \exists z \in L (z \sqsubseteq y \land x \sqsubseteq z), \\
\text{(L4)} & \quad x \in L \land x \ll y \implies \exists z \in L (x \ll z \ll y).
\end{align*}
\]

The class of the aforementioned structures will be denoted by \( \mathcal{R} \) and its elements will be called \textit{Roepers connection structures}.

In every structure from \( \mathcal{R} \) a filter \( \mathcal{F} \) is \textit{limited} iff \( \mathcal{F} \cap L \neq \emptyset \). Two filters are connected \( (\mathcal{F}_1 \bowtie \mathcal{F}_2) \) iff \( \forall x \in \mathcal{F}_1 \forall y \in \mathcal{F}_2 x \sqsubseteq y \). As it is proven in [7] the relation \( \bowtie \) is equivalence relation in the set of all limited ultrafilters of a given Rooper connection structure. This time points are defined as equivalence classes of \( \bowtie \) in the set Ult\(_l\)(\( \mathcal{M} \)) of all limited ultrafilters of a structure \( \mathcal{M} \in \mathcal{R} \):

\[
\Pi_\bowtie := \left[ \text{Ult}\_l(\mathcal{M}) \right]_{\bowtie}. \quad \text{(df } \Pi_\bowtie\text{)}
\]

Internal points of regions are defined in the following way:

\[
\text{Ir}_{\bowtie}(x) := \{ \alpha \in \Pi_\bowtie \mid x \in \bigcap \alpha \}. \quad \text{(df } \text{Ir}_{\bowtie}\text{)}
\]

Similarly as in case of structures from \( \mathcal{G} \), the set \{Ir\(_{\bowtie}(x) \mid x \in M\} \) for a structure \( \mathcal{M} \in \mathcal{R} \) is a basis, and thus \( (\Pi_\bowtie, \text{O}) \) with \( \text{O} \) defined in the standard way is a topological space.

3 Comparison

In our talk we aim at:

- comparing both constructions of points
- comparing properties of topological spaces generated by means of structures from \( \mathcal{G} \) and \( \mathcal{R} \).

References