On two ways of generating topological spaces from Grzegorczyk mereological structures

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1 Grzegorczyk connection structures

The first way was first described in [4] and was later presented and analyzed in [2] and [3]. It is a theory of triples $\langle M, \sqsubseteq, \mathbb{C}\rangle$ (with $\sqsubseteq$ being called *part of* relation and $\mathbb{C}$ – *connection* relation) such that $\langle M, \sqsubseteq\rangle$ is a poset with unity which satisfies the following axioms:

$$x \not\sqsubseteq y \longrightarrow \exists_{z\in M}(z \sqsubseteq x \wedge z \perp y \ \wedge \forall_{u\in M}(u \sqsubseteq x \wedge u \perp y \longrightarrow u \sqsubseteq z))\,, \tag{SSP+}$$

$$\forall_{x,y\in M}\exists_{u\in M}\, u \ \mathsf{Sup}\ \{x,y\}\,, \tag{$\exists$Sup}$$

$$\forall_{x,y\in M}\,(x \bigcirc y \longrightarrow \exists_{u\in M}\, u \ \mathsf{Inf}\ \{x,y\})\,. \tag{$\exists$Inf}$$

where:

$$x \sqsubset y \stackrel{\text{df}}{\longleftrightarrow} x \sqsubseteq y \wedge x \neq y\,, \tag{df $\sqsubset$}$$

$$x \bigcirc y \stackrel{\text{df}}{\longleftrightarrow} \exists_{z\in M}(z \sqsubseteq x \wedge z \sqsubseteq y)\,, \tag{df $\bigcirc$}$$

$$x \perp y \stackrel{\text{df}}{\longleftrightarrow} \neg\exists_{z\in M}(z \sqsubseteq x \wedge z \sqsubseteq y)\,. \tag{df $\perp$}$$

The above relations are called, respectively, *proper part*, *compatibility* and *incompatibility* relation. Sup and Inf are the standard *supremum* and *infimum* relations in posets. Every triple $\langle M, \sqsubseteq, \mathbb{C}\rangle$ satisfying the above set of postulates will be called *Grzegorczyk mereological structure*.[1]

The connection relation $\mathbb{C}$ is to satisfy the following three postulates:

$$\forall_{x,y\in R}\big(x \sqsubseteq y \longrightarrow x \,\mathbb{C}\, y\big)\,, \tag{C1}$$

$$\forall_{x,y\in R}\big(x \,\mathbb{C}\, y \longrightarrow y \,\mathbb{C}\, x\big)\,, \tag{C2}$$

$$\forall_{x,y,z\in R}\big(x \sqsubseteq y \wedge z \,\mathbb{C}\, x \longrightarrow z \,\mathbb{C}\, y\big)\,. \tag{C3}$$

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[1] The system was first presented in [5] in slightly different form. For a full presentation and analysis see [6].

In $\langle M, \sqsubseteq, \mathbb{C}\rangle$ we introduce binary relation of *non-tangential part*:

$$x \ll y \stackrel{\text{df}}{\longleftrightarrow} \forall_{z\in M}\,(z \perp y \longrightarrow \neg z \,\mathbb{C}\, x)\,. \qquad (\texttt{df} \ll)$$

By means of the auxiliary relation we define *a representative of a point*[2] as a set $\emptyset \neq Q \subseteq M$ such that:

$$\forall_{u,v\in Q}(u \neq v \longrightarrow u \ll v \vee v \ll u)\,, \qquad (\texttt{R1})$$

$$\forall_{u\in Q}\exists_{v\in X}\; v \ll u\,, \qquad (\texttt{R2})$$

$$\forall_{u,v\in R}\big(\forall_{z\in Q}(z \bigcirc u \wedge z \bigcirc v) \longrightarrow u \,\mathbb{C}\, v\big)\,. \qquad (\texttt{R3})$$

The set of representatives of points will be denoted by '**Q**'. We now introduce the so called *Grzegorczyk axiom*:

$$x \,\mathbb{C}\, y \longrightarrow \exists_{Q\in\mathbf{Q}}\,\big((x \bigcirc y \longrightarrow \exists_{z\in Q}\, z \sqsubseteq x \sqcap y) \wedge \forall_{z\in Q}\,(z \bigcirc x \wedge z \bigcirc y)\big)\,, \qquad (\texttt{G})$$

and call every Grzegorczyk mereological structure satisfying axioms (C1)–(C3), (G) *Grzegorczyk connection structure*. The class of all such structures will be denoted by '**G**'.

In every structure from **G** we define *points* as filters generated by representatives of points:

$$\Pi := \big\{X \subseteq M \mid \exists_{Q\in\mathbf{Q}}\, X = \{x \in M \mid \exists_{y\in Q}\, y \sqsubseteq x\}\big\}\,. \qquad (\texttt{df}\,\Pi)$$

Elements of Π will be denoted by small Greek letters: 'α', 'β', 'γ', 'δ'. With every element of the domain of a structure from the class **G** we associate the set of its *internal* points:

$$\mathbb{I}\mathbb{r}\mathbb{I}(x) := \{\alpha \in \Pi \mid x \in \alpha\}\,. \qquad (\texttt{df}\,\mathbb{I}\mathbb{r}\mathbb{I})$$

The set $\{\mathbb{I}\mathbb{r}\mathbb{I}(x) \mid x \in M\}$ is a basis, therefore the pair $\langle \Pi, \mathscr{O}\rangle$ where:

$$A \in \mathscr{O} \longleftrightarrow \exists_{\mathcal{A}\subseteq\mathscr{O}}\, A = \bigcup \mathcal{A}$$

is a topological space.

2 Roeper connection structures

The second way of constructing points and topological spaces originates from the ideas presented in [7]. This time structures we focus upon are of the form $\langle M, \mathbb{L}, \sqsubseteq, \mathbb{C}\rangle$ where:

- $\langle M, \sqsubseteq\rangle$ is a Grzegorczyk mereological structure,
- $\mathbb{C}$ satisfies (C1)–(C3) plus the following:

$$\forall_{x,y,z\in R}\big(x \,\mathbb{C}\, y \sqcup z \longrightarrow x \,\mathbb{C}\, z \vee x \,\mathbb{C}\, y\big)\,, \qquad (\texttt{C4})$$

[2] Grzegorczyk's representative of points are very similar to Whitehead's more known abstractive sets from [8].

– $\mathbb{L}$ is a subset of M whose elements are called *limited regions* and satisfy the following postulates:

$$x \in \mathbb{L} \wedge y \sqsubseteq x \longrightarrow y \in \mathbb{L}\,, \tag{L1}$$

$$x \in \mathbb{L} \vee y \in \mathbb{L} \longrightarrow x \sqcup y \in \mathbb{L}\,, \tag{L2}$$

$$x \mathrel{\mathbb{C}} y \longrightarrow \exists_{z\in\mathbb{L}}\,(z \sqsubseteq y \wedge x \mathrel{\mathbb{C}} z)\,, \tag{L3}$$

$$x \in \mathbb{L} \wedge x \ll y \longrightarrow \exists_{z\in\mathbb{L}}\, x \ll z \ll y\,. \tag{L4}$$

The class of the aforementioned structures will be denoted by '**R**' and its elements will be called *Roeper connection structures*.

In every structure from **R** a filter $\mathscr{F}$ is *limited* iff $\mathscr{F} \cap \mathbb{L} \neq \emptyset$. Two filters are connected ($\mathscr{F}_1 \infty \mathscr{F}_2$) iff $\forall_{x\in\mathscr{F}_1}\forall_{y\in\mathscr{F}_2}\, x \mathrel{\mathbb{C}} y$. As it is proven in [7] the relation ∞ is equivalence relation in the set of all limited ultrafilters of a given Roeper connection structure. This time points are defined as equivalence classes of ∞ in the set $\mathrm{Ult}_\ell(\mathfrak{M})$ of all limited ultrafilters of a structure $\mathfrak{M} \in \mathbf{R}$:

$$\Pi_{\mathtt{R}} := [\mathrm{Ult}_\ell(\mathfrak{M})]_\infty\,. \tag{df $\Pi_{\mathtt{R}}$}$$

Internal points of regions are defined in the following way:

$$\mathbb{Ir}_{\mathtt{R}}(x) := \{\alpha \in \Pi_{\mathtt{R}} \mid x \in \textstyle\bigcap \alpha\}\,. \tag{df $\mathbb{Ir}_{\mathtt{R}}$}$$

Similarly as in case of structures from **G**, the set $\{\mathbb{Ir}_{\mathtt{R}}(x) \mid x \in M\}$ for a structure $\mathfrak{M} \in \mathbf{R}$ is a basis, and thus $\langle \Pi_{\mathtt{R}}, \mathscr{O}\rangle$ with $\mathscr{O}$ defined in the standard way is a topological space.

3 Comparison

In our talk we aim at:

– comparing both constructions of points
– comparing properties of topological spaces generated by means of structures from **G** and **R**.

References

1. Bennet, B., and I. Düntsch, "Axioms, algebras and topology", pp. 99–159 in: *Handbook of Spatial Logics*, M. Aiello, I. Pratt-Hartmann, and J. Van Benthem (eds.), Springer, 2007.
2. Loredana Biacino, Giangiacomo Gerla. "Connection structures: Grzegorczyk's and Whitehead's definitions of point", *Notre Dame Journal of Formal Logic*, 37(3): 431–439, 1996.
3. Gruszczyński R., and A. Pietruszczak, "Space, points, mereology. On foundations of point-free Euclidean geometry", *Logic and Logical Philosophy*, vol. 18(2): 145–188, 2009.
4. Grzegorczyk, A., "Axiomatizability of geometry without points", *Synthese*, 12(2–3): 228–235, 1960.

5. Grzegorczyk, A., “The systems of Leśniewski in relation to contemporary logical research”, *Studia Logica* 3 (1):77–95, 1955.
6. Pietruszczak, A., *Podstawy teorii części*, Toruń, 2013.
7. Roeper, P., “Region-based topology”, *Journal of Philosophical Logic*, Vol. 26, no. 3, pp. 251–309, 1997.
8. Whitehead, A.N., *Process and reality*, Free Press, 1979.