Distributive Contact Lattices with Nontangential Part-of Relations

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Abstract. In a Boolean algebra, the "contact" and "nontangential part-of" relations are linked and interdefinable thanks to the complement. These relations yield on the Stone dual of the algebra two closed relations that appear to be the same. In a distributive lattice, the contact and nontangential part-of relations also yield dual closed relations on the Priestley dual. However, as the link between them is lost, these relations are not equal anymore. We explore the conditions linking the contact and nontangential part-of relations one should add in order to recover these relations knowing the intersection of their dual.

1 Introduction

This talk falls within the scope of region-based theory of space. The extension of topological operators has already been largely considered ; however, these operators were always studied alone, and "simultaneous" studies of these operators were asked in [2]. The aim of this talk is to extend the boolean notions of "contact" and "non-tangential part-of" relations to the distributive lattice case.

As the two relations arise from a common concept, it is natural to desire that they stay linked in the generalisation. It would be natural to define a distributive contact lattice with a nontangential part-of relation as a distributive lattice endowed with two binary relations that can be embedded in a Boolean contact algebra. This comes down to require that the two dual relations can be recovered knowing their intersection. This is thus essentially the property we are trying to obtain.

2 Boolean contact algebras

A Boolean contact algebra is a Boolean algebra $(B, \vee, \wedge, 0, 1, ^c)$ endowed with a binary relation $\mathcal{C}$ satisfying the following axioms.

C0 $0 \not\mathcal{C}\, a$, $a \not\mathcal{C}\, 0$
C1 $0 \neq a \leq b, c \Rightarrow b \,\mathcal{C}\, c$
C2 $a \,\mathcal{C}\, b \Rightarrow b \,\mathcal{C}\, a$
C3 $a \geq b \,\mathcal{C}\, c \leq d \Rightarrow a \,\mathcal{C}\, d$
C4 $(a \vee b) \,\mathcal{C}\, (c \vee d) \Rightarrow a \,\mathcal{C}\, c$ or $a \,\mathcal{C}\, d$ or $b \,\mathcal{C}\, c$ or $b \,\mathcal{C}\, d$

One can then define the nontangential part-of relation $\prec$ by $a \prec b \Leftrightarrow a \not\mathcal{C}\, b^c$. This relation satisfies the following corresponding axioms.

DV0 $0 \prec a \prec 1$
DV1 $a \prec b \Rightarrow a \leq b$
DV2 $a \prec b \Rightarrow b^c \prec a^c$
DV3 $a \leq b \prec c \leq d \Rightarrow a \prec d$
DV4 $a, c \prec b, d \Rightarrow a \vee c \prec b \wedge d$

These two relations are per se linked by the complement. Each of these relations yields a closed relation on the Stone dual of B, which appear to be the same.

3 Extension to distributive lattices

We now consider a distributive lattice D endowed with two relations $\mathcal{C}$ and $\prec$ satisfying C0, C3, C4 and DV0, DV3, DV4. We are trying to keep as few axioms as possible so that the considered contact lattices extend not only the Boolean contact algebras, but also the Boolean algebras with operators. Dropping axioms such as symmetry is natural when trying to modelize time for example (see also [1]).

The relations $\prec$ and $\mathcal{C}$ can be captured by the maps $f, g : D \to \mathcal{P}(D)$ defined by

$$f(a) = \{b \in D : a \prec b\} \tag{1}$$

and

$$g(a) = \{b \in D : a \not\mathcal{C}\, b\} \ . \tag{2}$$

The sets $f(a)$ and $g(a)$ are respectively a filter and an ideal, and the maps f and g are some kind of "multi-operators", that is $f(0) = D$, $f(a \vee b) = f(a) \cap f(b)$, $g(0) = D$ and $g(a \vee b) = g(a) \cap g(b)$. These "multi-operators" yield closed relations on the Priestley dual space of D defined by

$$x \, R_f \, y \Longleftrightarrow \bigcup f(F_x) \subset F_y \tag{3}$$

and

$$x \, S_g \, y \Longleftrightarrow \bigcup g(F_x) \subset I_y \ , \tag{4}$$

which satisfy $\geq \circ R_f \circ \geq = R_f$ and $\geq \circ S_g \circ \leq = S_g$.

The central question is : which axioms should one add to C0–3 and DV0–3 in order to be able to recover R_f and S_g knowing

$$T = R_f \cap S_g \ . \tag{5}$$

The corresponding problem for the modal operators $\Box$ and $\Diamond$ has already been examined in [5] where it is showed that the condition cannot be expressed with any system of quasiequations. In our case, we show that it can be translated as the axioms

CDV1 $a \prec b \vee c, a \not{C} c \Rightarrow a \prec b$

and

CDV2 $a \not{C} b \wedge c, a \prec b \Rightarrow a \not{C} c$

and further investigate the properties of these relations and their duals.

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