Hereditary Structural Completeness in Intermediate Logics

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If $\vdash$ is a (finitary, structural, single-conclusion) consequence relation (in a given propositional language), by $\mathsf{Th}(\vdash)$ we denote the set of the theorems of $\vdash$. Recall that a consequence relation $\vdash$ is called *structurally complete* ($\mathcal{SC}$ for short), if $\vdash\ \subset\ \vdash'$ yields $\mathsf{Th}(\vdash) \subset \mathsf{Th}(\vdash')$ for any consequence relation $\vdash'$ extending $\vdash$. And $\vdash$ is *hereditarily structurally complete* ($\mathcal{HSC}$ for short) if $\vdash$ and all its extensions are $\mathcal{SC}$. For every consequence relation $\vdash$ there is the greatest consequence relation $\vdash^\circ$ that has the same set of theorems as $\vdash$. Clearly, $\vdash^\circ$ is $\mathcal{SC}$, and we call this consequence relation *a structural completion of* $\vdash$. If L is a logic (understood as a set of formulas closed under modus ponens and substitutions), by L° we denote a structural completion of L, that is the greatest consequence relation having L as its set of theorems. For instance, Int° is a consequence relation defined by axiom schemata of intuitionistic propositional logic, modus ponens and Visser rules (for definitions cf. [6]).

If R is a set of (finitary structural single-conclusion) rules, we say that rules R are *admissible* for $\vdash$ if $\mathsf{Th}(\vdash) = \mathsf{Th}(\vdash^{\mathsf{R}})$, where $\vdash^{\mathsf{R}}$ is the least consequence relation containing $\vdash$ and all rules from R.

Proposition 1. *Let $\vdash$ be a consequence relation and R be the set of all rules admissible for $\vdash$. Then $\vdash^\circ$ is $\mathcal{HSC}$ if and only if rules R form a basis of admissible rules in every extension of $\vdash$ where these rules are admissible.*

Proof (a sketch). Let $\vdash$ be a consequence relation and R be a set of all rules admissible for $\vdash$. Then $\vdash^\circ = \vdash^{\mathsf{R}}$. Suppose rules R form a basis of admissible rules of a consequence relation $\vdash_0$ extending $\vdash^\circ$. Since all rules from R are $\vdash$-derivable and $\vdash_0$ is an extension of $\vdash^\circ$, all rules R are $\vdash_0$-derivable. By assumption, rules R form a basis of rules admissible for $\vdash_0$. Hence, all admissible for $\vdash_0$ rules are $\vdash_0$-derivable, that is, $\vdash_0$ is $\mathcal{SC}$.

Conversely, suppose $\vdash^\circ$ is $\mathcal{HSC}$ consequence relation and R is the set of all rules admissible for $\vdash^\circ$. Let $\vdash_0$ be a consequence relation extending $\vdash^\circ$. Then all rules admissible for $\vdash_0$ are $\vdash_0$-derivable: if there is a rule r admissible for $\vdash_0$ but not $\vdash_0$-derivable, we would have $\vdash_0\ \subset\ \vdash_0^{\{r\}}\ \subseteq\ \vdash_0^\circ$, i.e. $\vdash_0$ would be not $\mathcal{SC}$, and this would contradict the assumption that $\vdash^\circ$ is $\mathcal{HSC}$.

From [6, Theorem 3.9] and Proposition 1 we get the following:

Corollary 1. Int° *is hereditarily structurally complete. Hence, the following consequence relations are $\mathcal{HSC}$:* $\mathsf{KC}^\circ, \mathsf{M_n}^\circ, \mathsf{BD_1}^\circ, \mathsf{G_k}^\circ, \mathsf{LC}^\circ, \mathsf{Sm}^\circ, \mathsf{V}^\circ$.

Recall from [4] that the set of all $\mathcal{HSC}$ intermediate logics forms a countable principal filter of the lattice of all intermediate logics, and every $\mathcal{HSC}$ intermediate logic is finitely axiomatizable. The situation with $\mathcal{HSC}$ structural completions is totally different. In fact, in [8] it was proven that there is continuum many intermediate logics admitting Visser rules. Hence, the following holds.

Theorem 1. *(a) There is continuum many intermediate logics having $\mathcal{HSC}$ structural completion*
(b) There are not finitely axiomatizable intermediate logics having $\mathcal{HSC}$ structural completion
(c) There are not finitely approximated $\mathcal{HSC}$ consequence relations
(d) The class of all $\mathcal{HSC}$ consequence relations has no least element, thus, this class does not form a lattice (or even a lower semilattice).

For algebraizable (in sense of [1]) logics, every consequence relation $\vdash$ has a corresponding quasivariety $\mathcal{Q}_{\vdash}$, and $\vdash$ is $\mathcal{SC}$ if and only if $\mathcal{Q}_{\vdash}$ is generated by its free algebra $\mathbf{F}_{\omega}(\mathcal{Q}_{\vdash})$ of a countable rank (see e.g. [7, 2]). If $\mathcal{Q}$ is a quasivariety, by $\mathcal{Q}^{\circ}$ we denote the least quasivariety generating the same variety as $\mathcal{Q}$.

If $\mathbf{A}$ is an algebra, by $\mathcal{Q}(\mathbf{A})$ we denote a quasivariety generated by $\mathbf{A}$. If $\mathcal{Q}$ is a quasivariety and θ is a congruence of algebra $\mathbf{A}$, we say that θ is a *$\mathcal{Q}$-congruence* if $\mathbf{A}/\theta \in \mathcal{Q}$.

A quasivariety $\mathcal{Q}$ is said to be *primitive* if every subquasivariety of $\mathcal{Q}$ is structurally complete (see [7, 2]).

Given a quasivariety $\mathcal{Q}$, an algebra $\mathbf{A}$ is called *weakly $\mathcal{Q}$-projective* if $\mathbf{A}$ is embedded in its every homomorphic preimage from $\mathcal{Q}$; and $\mathbf{A}$ is called *$\mathcal{Q}$-irreducible* if the meet of all proper $\mathcal{Q}$-congruences of $\mathbf{A}$ is a proper $\mathcal{Q}$-congruence.

Recall from [5] that a locally finite quasivariety $\mathcal{Q}$ is primitive if and only if all its finitely generated $\mathcal{Q}$-irreducible algebras are weakly $\mathcal{Q}$-projective.

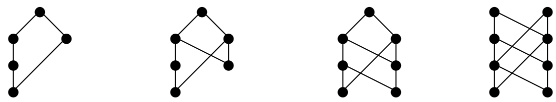
Denote by Z_k a k-element single-generated Heyting algebra, and by Z - the infinite single-generated algebra – the Rieger-Nishimura ladder. Let $\mathcal{H}_n$ denotes a variety of all Heyting algebras of height n. If $\mathbf{A}, \mathbf{B}$ are Heyting algebras, by $\mathbf{A} \oplus \mathbf{B}$ we denote a concatenation of $\mathbf{A}$ and $\mathbf{B}$, that is, $\mathbf{A} \oplus \mathbf{B}$ is the algebra obtained by putting $\mathbf{B}$ on top of $\mathbf{A}$ and identifying the greatest element of $\mathbf{A}$ with the least element of $\mathbf{B}$.

The Proposition below follows from [3] and [6, Theorem 5.4].

Proposition 2. *(a) Every finitely generated weakly $\mathcal{H}_n^{\circ}$-projective algebra is of shape $\mathbf{A} \oplus \mathbf{B}$, where $\mathbf{A} \oplus \mathsf{Z}_2$ is a projective Heyting algebra.*
(b) a finitely generated s.i. algebra is weakly $\mathcal{H}_n^{\circ}$-projective if and only if it is a projective Heyting algebra;
(c) finitely generated not s.i. algebras of the height less than n are not weakly $\mathcal{H}_n^{\circ}$-projective.

From the above proposition we obtain the following theorem.

Theorem 2. *(a) $\mathcal{Q}(\mathsf{Z})$ is primitive;*
(b) $\mathcal{Q}(\mathsf{Z}_{2k+1})$ is primitive if and only if $k \in \{1, 2, 4\}$;
(c) $\mathcal{Q}(\mathsf{Z}_{2k)}$ is primitive if and only if $k < 8$.



$Z_2 \oplus Z_7$ $Z_2 \oplus Z_{10}$ $Z_2 \oplus Z_{12}$ Z_{16}

Fig.1

Indeed, if $k \notin \{1, 2, 4\}$, algebra Z_{2k+1} is free in $\mathcal{Q}_k := \mathcal{Q}(Z_{2k+1})$. Observe that for all $k \notin \{1, 2, 4\}$ algebra $Z_2 \oplus Z_7$ (see Fig.1) is embedded into Z_{2k+1}, hence $Z_2 \oplus Z_7 \in \mathcal{Q}_k$. Algebra $Z_2 \oplus Z_{10} \in \mathcal{Q}_k$, for $Z_2 \oplus Z_{10}$ is a subdirect product of algebras $Z_{\oplus} Z_7$ and $Z_2 \oplus Z_5$, and the latter algebra is embedded into Z_{2k+1}. But algebra $Z_2 \oplus Z_7$ is not weakly $\mathcal{Q}_k$-projective: algebra $Z_2 \oplus Z_7$ is a homomorphic image of $Z_2 \oplus Z_{10}$, but $Z_2 \oplus Z_7$ is not embedded into $Z_2 \oplus Z_{10}$.

If $k \geq 8$, the quasivariety $\mathcal{Q}_k := \mathcal{Q}(Z_{2k})$ is not primitive for the following reason (we consider case $k = 8$): algebra Z_{16} is free in $\mathcal{Q}_8$, algebras $Z_2 \oplus Z_{10}$ and $Z_2 \oplus Z_{12}$ are embedded in Z_{16} and, hence, $Z_2 \oplus Z_{10}, Z_2 \oplus Z_{12} \in \mathcal{Q}_8$; algebra $Z_2 \oplus Z_{10}$ is $\mathcal{Q}_8$-irreducible, but not weakly $\mathcal{Q}_8$-projective (algebra $Z_2 \oplus Z_{10}$ is a homomorphic image of $Z_2 \oplus Z_{12}$, but not embedded into the latter).

Recall that Z_{4n} is a single-generated free algebra of $\mathcal{H}_n$. Since $\mathcal{Q}(Z_{4n})$ is not primitive for all $n \geq 4$, we can conclude the following.

Corollary 2. *For every $n \geq 4$ the quasivariety $\mathcal{H}_n^{\circ}$ is not primitive. In other words, the structural completions of logics BD_n are not $\mathcal{HSC}$ for all $n \geq 4$.*

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