A categorical structure of realizers for the Minimalist Foundation

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We provide a categorical presentation of a realizability interpretation à la Kleene for the Minimalist Foundation. This interpretation is predicative since it is made in Feferman's predicative theory of inductive definitions $\widehat{ID_1}$. Our final aim is to provide a candidate for a predicative version of Hyland's effective topos [1].

The Minimalist Foundation (for short **MF**) is a predicative foundation for constructive mathematics. It was ideated by M. E. Maietti and G. Sambin in [7] and then completed in [3] by M. E. Maietti. It is called *minimalist* since it is intended to constitute a common core among the most relevant constructive and classical foundations, introduced both in type theory, in category theory and in axiomatic set theory.

One of its novelties is that it consists of two levels: an intensional level (**mTT**) which should be a (type) theory with enough decidable properties to be a base for a proof-assistant and for extraction of computational contents from its proofs, and an extensional level (**emTT**) formulated in a language as close as possible to that of ordinary mathematics. Both the intensional level and the extensional level of the Minimalist Foundation consists of dependent type systems based on versions of Martin-Löf's type theory: the intensional one is based on [9] and the extensional one on [8].

The presence of two levels is also relevant to show the compatibility of **MF** both with intensional theories as those formulated in type theory, such as Martin-Löf's type theory or Coquand's Calculus of Constructions, and with extensional theories as those formulated in axiomatic set theory, such as Aczel's Constructive Zermelo-Fraenkel set theory, or those formulated in category theory, such as topoi or pretopoi.

In order to extract programs from proofs done in (the extensional level of) the Minimalist Foundation in [4] we provided an intepretation of the intensional level **mTT** of MF into Feferman's predicative theory of inductive definitions $\widehat{ID_1}$. We followed Beeson's interpretation for first order Martin-Löf's type theory [9] with one universe in $\widehat{ID_1}$ to interpret **mTT**-sets and collections by using fix-points when necessary. Our interpretation differs from Beeson's one for the fact that we interpret **mTT** propositions in a proof-irrelevant way in order to make the extended Formal Church thesis (for short CT), valid in our interpretation. Beeson's model can not validate CT because it validates the axiom of choice and extensional equality of functions which are inconsistent with CT.

Here we simplify the interpretation given in [4] by placing it in a categorical model. In more detail we pass from an untyped interpretation with involved substitution lemmas, to a partial typed interpretation à la Streicher as in [10] for which substitution lemmas follow in a clearer and simpler way. This is because we adopt indexed categories where the categorical composition turns out to be interpreted as a substitution in the categorical model based on $\widehat{ID_1}$. In particular we adopt four indexed categories: one to interpret **mTT**-sets, one to interpret **mTT**-collections, one to interpret **mTT**-propositions and one to interpret **mTT**-small propositions (i.e. with quantifiers restricted to sets).

Then on the resulting categorical structure, we employ the notion of elementary quotient completion introduced in [6], [5] to build a quotient model for the extensional level **emTT** of **MF** with (extended) CT.

We consider such a quotient model as a candidate for a predicative effective topos. Actually, since **emTT** can be naturally interpreted in the internal type theory of a topos with a Natural Numbers object in [2], also Hyland's effective topos easily provides a model for the extensional level **emTT** of **MF** and shows its consistency with CT, but, of course, in an impredicative theory.

We leave to future work to compare our effective quotient model based on $\widehat{ID_1}$ with Hyland's effective topos.

Keywords: realizability, type theory, indexed category

References

1. J. M. E. Hyland. The effective topos. In A. S. Troesltra and D. van Dalen, editors, *The L. E. J. Brouwer centenary symposium*, Studies in logic and the foundations of mathematics, pages 165–216. North-Holland, 1982.
2. M. E. Maietti. Modular correspondence between dependent type theories and categories including pretopoi and topoi. *Mathematical Structures in Computer Science*, 15:1089–1149, 2005.
3. M. E. Maietti. A minimalist two-level foundation for constructive mathematics. *Annals of Pure and Applied Logic*, 160(3):319–354, 2009.
4. M. E. Maietti and S. Maschio. An extensional Kleene realizability model for the Minimalist Foundation. *submitted, http://arxiv.org/abs/1502.02864*, 2014.
5. M. E. Maietti and G. Rosolini. Elementary quotient completion. *Theory and Applications of Categories*, 27(17):445–463, 2013.
6. M. E. Maietti and G. Rosolini. Quotient completion for the foundation of constructive mathematics. *Logica Universalis*, 7(3):371–402, 2013.
7. M. E. Maietti and G. Sambin. Toward a minimalist foundation for constructive mathematics. In L. Crosilla and P. Schuster, editor, *From Sets and Types to Topology and Analysis: Practicable Foundations for Constructive Mathematics*, number 48 in Oxford Logic Guides, pages 91–114. Oxford University Press, 2005.
8. P. Martin-Löf. *Intuitionistic Type Theory. Notes by G. Sambin of a series of lectures given in Padua, June 1980.* Bibliopolis, Naples, 1984.
9. B. Nordström, K. Petersson, and J. Smith. *Programming in Martin Löf's Type Theory.* Clarendon Press, Oxford, 1990.
10. Th. Streicher. *Semantics of type theory.* Birkhäuser, 1991.