Stable modal logics

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Stable superintuitionistic logics were introduced in [1], and further studied in [3]. Stable modal logics were introduced in [2]. Stable modal logic form a large class of normal modal logics with good properties such as the finite model property (FMP). Here we continue the study of stable modal logics.

Let $\mathfrak{A} = (A, \Diamond)$ and $\mathfrak{B} = (B, \Diamond)$ be modal algebras. We recall [2] that a Boolean homomorphism $h : A \to B$ is *stable* provided $\Diamond h(a) \leq h(\Diamond a)$ for all $a \in A$; and that $\mathfrak{B}$ is a *stable subalgebra* of $\mathfrak{A}$ if B is a Boolean subalgebra of A and the inclusion $B \hookrightarrow A$ is a stable embedding. A class $\mathcal{K}$ of modal algebras is *stable* if whenever $\mathfrak{B}$ is isomorphic to a stable subalgebra of $\mathfrak{A}$ and $\mathfrak{A} \in \mathcal{K}$, then $\mathfrak{B} \in \mathcal{K}$.

Definition 1. *We call a normal modal logic L* stable *if the variety $\mathcal{V}(L)$ corresponding to L is generated by a stable universal class.*

Definition 2. *Let $\mathfrak{A} = (A, \Diamond)$ be a finite modal algebra. For every $a \in A$, let p_a be a propositional letter. The* stable multi-conclusion rule $\rho(\mathfrak{A})$ associated with $\mathfrak{A}$ *is defined as Γ/Δ, where*

$$\begin{aligned} \Gamma = &\{p_{a\vee b} \leftrightarrow p_a \vee p_b \mid a, b \in A\} \cup \\ &\{p_{\neg a} \leftrightarrow \neg p_a \mid a \in A\} \cup \\ &\{\Diamond p_a \to p_{\Diamond a} \mid a \in A\} \end{aligned}$$

and

$$\Delta = \{p_a \mid a \in A, a \neq 1\}.$$

Theorem 1. *[2] A normal modal logic L is stable iff L is axiomatizable by stable multi-conclusion rules.*

Using duality between finite modal algebras and finite Kripke frames, we often talk about stable rules of finite Kripke frames. When drawing Kripke frames, we use the standard convention that $\bullet$ depicts an irreflexive point, while $\circ$ depicts a reflexive point. Below are several examples of stable normal modal logics.

Example 1. Recall that

KD $=$ **K** $+ \Box p \to \Diamond p$ is the logic of serial frames;
KT $=$ **K** $+ p \to \Diamond p$ is the logic of reflexive frames.

These logics are stable. In fact,

KD is axiomatizable by $\rho(\bullet)$;
KT is axiomatizable by $\rho(\bullet)$ and $\rho(\bullet\!\longleftrightarrow\!\circ)$.

On the other hand, **K4** is not a stable logic. Therefore, we slightly modify the concept of a stable logic for normal extensions of **K4**.

Definition 3.
1. *We call a class $\mathcal{K}$ of* **K4***-algebras* **K4**-stable *if whenever $\mathfrak{A} \in \mathcal{K}$ and $\mathfrak{B}$ is a* **K4***-algebra that is isomorphic to a stable subalgebra of $\mathfrak{A}$, then $\mathfrak{B} \in \mathcal{K}$.*
2. *Let L be a normal extension of* **K4***. We call L* **K4**-stable *if the variety $\mathcal{V}(L)$ is generated by a* **K4***-stable universal class.*

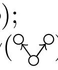
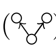
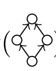
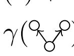
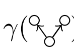
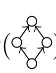
The next theorem gives a characterization of **K4**-stable logics. For a **K4**-algebra $\mathfrak{A} = (A, \Diamond)$, recall that $\Diamond^+ a := a \vee \Diamond a$. We call $\mathfrak{A}$ *well-connected* if $\Diamond^+ a \wedge \Diamond^+ b = 0$ implies $a = 0$ or $b = 0$. If $\mathfrak{A}$ is a finite well-connected **K4**-algebra, then we associate with $\mathfrak{A}$ the *stable formula* $\gamma(\mathfrak{A})$, where

$$\gamma(\mathfrak{A}) := \bigwedge\{\Box^+\gamma \mid \gamma \in \Gamma\} \to \bigvee\{\Box^+\delta \mid \delta \in \Delta\}$$

and Γ, Δ are as in Definition 2.

Theorem 2. *Let L be a normal extension of* **K4***. Then L is* **K4***-stable iff the class of well-connected members of $\mathcal{V}(L)$ is* **K4***-stable. In fact, each* **K4***-stable logic is axiomatizable by stable formulas.*

Below are some well-known normal extensions of **K4** that are **K4**-stable. We also list the stable formulas that axiomatize them.

Example 2.
1. **D4** = **K4** + $\gamma(\bullet)$;
2. **K4.2** = **K4** + $\gamma($$)$;
3. **K4.3** = **K4** + $\gamma($$)$ + $\gamma($$)$;
4. **K4B** = **K4** + $\gamma($$)$;
5. **S4** = **K4** + $\gamma(\bullet)$ + $\gamma($$)$;
6. **S4.2** = **S4** + $\gamma($$)$;
7. **S4.3** = **S4** + $\gamma($$)$ + $\gamma($$)$;
8. **S5** = **S4** + $\gamma($$)$.

There exist normal extensions of **K4** that are not **K4**-stable, but are still axiomatized by stable formulas. However, among normal extensions of **S4**, being stable is equivalent to being axiomatized by stable formulas.

Theorem 3.
1. *There are continuum many non-transitive stable modal logics.*
2. *There are continuum many* **K4***-stable logics between* **K4** *and* **S4**.
3. *There are continuum many* **K4***-stable logics above* **S4**.

References

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