Constructions of Pretoposes

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Abstract. We explore certain aspects of the connection between regular categories and pretoposes, recalling in particular the construction of the pretopos freely generated by a regular category. This is applied to a conceptual completeness type of result in regular logic. Further, we analyze the factorization of a morphism of pretoposes as a quotient followed by a conservative morphism in terms of Benabou's regular congruences.

Keywords: pretopos, regular category, exact category, completion, regular congruence, quotient - conservative factorization, conceptual completeness, regular logic

1 Coproduct completion of regular categories

It is well-known that the construction of the pretopos associated with a coherent category takes place in two steps. First, one forms the completion of the coherent category under coproducts, then one takes the exact completion of the latter as a regular category (the *ex/reg* completion, [4]). Certain aspects of this construction have not been stressed out in the literature, mostly because the ingredients of these two steps had not been analyzed systematically by the time the pretopos associated with a coherent category was introduced in categorical logic.

The following proposition generalizes slightly [3], A1.4.5:

Proposition 1. *If $\mathcal{C}$ is a regular category, then its free completion under sums,* Fam$\mathcal{C}$*, is also regular.*

Proof: The crucial ingredient in the proof (going beyond the proof in loc. cit., which relies on the existence of suprema of subobjects) is the characterization of strong epimorphisms $(\alpha, f_i)\colon (C_i)_{i\in I} \to (C'_j)_{j\in J}$ in Fam$\mathcal{C}$, as those arrows given by a surjection $\alpha\colon I \to J$ such that each $f_i\colon C_i \to C'_{\alpha(i)}$ is a strong epimorphism. ■

The conclusion of the above Proposition remains valid if we confine ourselves to the finite coproduct completion hence, with the aid of [5], Theorem 2.3, stating that the exact completion, as a regular category, of an extensive category is itself extensive, we find the pretopos associated with a coherent category $\mathcal{C}$

as $(\mathrm{fam}\mathcal{C})_{ex/reg}$. We also note that each step of the construction preserves the existence of right adjoints to pullbacks of subobjects, so that the completion under coproducts of a Heyting category is Heyting category, while the ex/reg completion of a Heyting extensive category is a Heyting pretopos ([9], 2.1.27).

The construction applied to general regular categories (rather than just coherent ones) provides a left biadjoint to the inclusion $PRETOP \hookrightarrow REG$ of the 2-category of small pretoposes into the 2-category of small regular categories. This left biadjoint can be exploited to obtain a constructive, "syntactical" proof of a conceptual completeness result for regular logic, implicit in the work M. Makkai in [7] and the more recent treatment of it in [2]. More precisely, if an interpretation between two regular theories $\mathbb{T}$, $\mathbb{T}'$, seen as a regular functor $I: \mathcal{C}_{\mathbb{T}} \to \mathcal{C}_{\mathbb{T}'}$ between their respective syntactical categories, induces an equivalence between their categories of models, then the respective categories, have equivalent exact completions as regular categories. (Note that both the aforementioned works rely on the existence of sufficiently many models for such theories.) In particular, from conceptual completeness for pretoposes, we obtain that their respective associated pretoposes are equivalent and then we conclude with the aid of Proposition 1.4.9 in [8] and the following two propositions.

Proposition 2. *For a regular functor $I: \mathcal{C} \to \mathcal{D}$, if the induced*

$$(\mathrm{fam}I)_{ex/reg}: (\mathrm{fam}\mathcal{C})_{ex/reg} \to (\mathrm{fam}\mathcal{D})_{ex/reg}$$

is fully faithful, then

$$I_{ex/reg}: \mathcal{C}_{ex/reg} \to \mathcal{D}_{ex/reg}$$

is fully faithful.

Proposition 3. *For a regular functor $I: \mathcal{C} \to \mathcal{D}$, if the induced*

$$(\mathrm{fam}I)_{ex/reg}: (\mathrm{fam}\mathcal{C})_{ex/reg} \to (\mathrm{fam}\mathcal{D})_{ex/reg}$$

is covering, then I (hence also $I_{ex/reg}: \mathcal{C}_{ex/reg} \to \mathcal{D}_{ex/reg}$) is covering.

2 Regular Congruences and the Quotient - Conservative Factorization

We analyze the factorization of a morphism of pretoposes as a quotient followed by a conservative morphism [6] in terms of Benabou's regular congruences [1]. We give a sufficient condition, in a category with coproducts, for a class of morphisms admitting a calculus of right fractions to yield a category of fractions that has itself coproducts:

Proposition 4. *Let $\mathcal{C}$ be a category with coproducts and Σ a class of morphisms of $\mathcal{C}$ admitting a calculus of right fractions. Assume that, for all $s: A \to B$ and $t: C \to D$ in Σ, we have that $s \sqcup t: A \sqcup C \to B \sqcup D$ is in Σ. Then the category of fractions $\mathcal{C}[\Sigma^{-1}]$ has coproducts and the quotient functor $P_{\Sigma}: \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ preserves them.*

Proof: Given objects A, B in $\mathcal{C}[\Sigma^{-1}]$, their coproduct is given by $A \sqcup B$, because if $A \xleftarrow{s} I \xrightarrow{f} C$ and $B \xleftarrow{t} J \xrightarrow{g} C$ are two arrows from A to C and from B to C, respectively, in $\mathcal{C}[\Sigma^{-1}]$, then $A \sqcup B \xleftarrow{s \sqcup t} I \sqcup J \xrightarrow{[f,g]} C$ is the required unique factorization through $A \sqcup B$ making the two triangles commutative. ■

In particular we have (using the terminology of [1])

Corollary 1. *If $F: \mathcal{C} \to \mathcal{D}$ is a morphism of pretoposes then the class* ker$F$ *of morphisms inverted by F, is a regular congruence that has the above property and the category of fractions $\mathcal{C}[\mathrm{ker}F^{-1}]$ is regular extensive.*

The passage from an exact category to a regular category of fractions does not necessarily preserve exactness: The coreflection to the inclusion of Stone spaces into the dual category of presheaves on finite Boolean algebras provides a counterexample. Hence the factorization, in the category of pretoposes, of F as a quotient functor followed by a conservative one arises via the ex/reg completion of $\mathcal{C}[\mathrm{ker}F^{-1}]$. In particular we obtain the description of quotient morphisms of pretoposes as those pretopos morphisms that are full on subobjects and subcovering and a more direct description of the mediating pretopos on the quotient-conservative factorization of a pretopos morphism (in contrast to 2.4.7 in [6]). We expect this to yield easier arguments for conceptual completeness and definability results in coherent logic.

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