Strict Implication Logics and Lambek Calculi

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A strict implication is an implication which is interpreted as the combination of material implication and a kind of necessity. Using modal logic, a strict impaction $\phi \to \psi$ can be translated as $\Box(\phi \supset \psi)$ where $\supset$ is the material implication in classical propositional logic. The strict implication language $\mathcal{L}_S$ consists of a denumerable set $\mathcal{V}$ of propositional variables, connectives $\wedge, \vee, \to$ and constants $\bot$. The set $\mathcal{L}_S$ of all formulas is defined inductively by the following rule:

$$\mathcal{L}_S \ni \phi ::= p \mid \bot \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \to \phi),$$

A sequent is an expression of the form $\Gamma \vdash \phi$ where Γ is a finite multiset of formulas. The strict implication fragment of modal logics are studied in [9, 5, 8, 4, 7, 3]. Here we concentrate on weak strict implication logics in [3] which axiomatize logical consequences over classes of frames. The minimal weak strict implication logic wK_σ consists of the following axioms and rules:

$$\text{(Id) } \phi \vdash \phi \quad \text{(Slly) } \phi \to \psi, \psi \to \chi \vdash \phi \to \psi$$

$$\text{(M1) } \phi \to \psi, \phi \to \chi \vdash \phi \to (\psi \wedge \chi) \quad \text{(M2) } \phi \to \chi, \psi \to \chi \vdash (\phi \vee \psi) \to \chi$$

$$\frac{\Gamma \vdash \phi}{\Gamma, \psi \vdash \phi}(w) \quad \frac{\Gamma \vdash \bot}{\Gamma \vdash \phi}(\bot\mathrm{R}) \quad \frac{\Gamma, \phi, \psi \vdash \delta}{\Gamma, \phi \wedge \psi \vdash \delta}(\wedge\mathrm{L}) \quad \frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}(\wedge\mathrm{R})$$

$$\frac{\Gamma, \phi \vdash \chi \quad \Gamma, \psi \vdash \chi}{\Gamma, \phi \vee \psi \vdash \chi}(\wedge\mathrm{L}) \quad \frac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \vee \psi_2}(\vee\mathrm{R}) \quad \frac{\phi \vdash \psi}{\emptyset \vdash \phi \to \psi}(\mathrm{DT}_0) \quad \frac{\Gamma \vdash \phi \quad \Gamma, \phi \vdash \psi}{\Gamma \vdash \psi}(\mathrm{cut})$$

Weak Heyting algebras are algebras for wK_σ which is the strict implication fragment of the minimal normal modal logic K. These algebras are studied in [6]. A *weak Heyting algebra* (WHA) is an algebra $(A, \wedge, \vee, \bot, \top, \to)$ where $(A, \wedge, \vee, \bot, \top)$ is a bounded distributive lattice and $\to$ is a binary operation on A satisfying the following conditions for all $a, b, c \in A$:

(C1) $(a \to b) \wedge (a \to c) = a \to (b \wedge c)$,
(C2) $(a \to c) \wedge (b \to c) = (a \vee b) \to c$,
(C3) $a \to a = \top$,
(C4) $(a \to b) \wedge (b \to c) \leq (a \to c)$,

where $\leq$ is the lattice order.

In this paper, we will consider those Lambek calculi into which strict implication logics can be conservatively extended. The idea behind this work is that strict implication algebras can be viewed as reducts of residuated groupoids. Residuated groupoids are algebras for Lambek calculi. For those Lambek calculi, we can construct Gentzen-style sequent calculi. By cut elimination and the subformula property for those sequent calculi, we can obtain natural sequent calculi for weak strict implication logics.

A *bounded distributive lattice-ordered residuated groupoid* (BDRG) is an algebra $(A, \wedge, \vee, \top, \bot, \to, \cdot, \leftarrow)$ where $(A, \wedge, \vee, \top, \bot)$ is a bounded distributive lattice and $\cdot, \to, \leftarrow$ are binary operations on A satisfying the following residuation condition for all $a, b, c \in A$: (RES) $a \cdot b \leq c$ iff $b \leq a \to c$ iff $a \leq c \leftarrow b$. For sequent calculus DFNL^+ for BDRGs, see e.g. [1].

Lemma 1. *Let $(A, \wedge, \vee, \top, \bot, \to, \cdot, \leftarrow)$ be a BDRG. Then its $(\wedge, \vee, \top, \bot, \to)$-reduct is a WHA iff the following conditions holds for all $a, b, c \in A$, (w^*) $a \cdot b \leq a$, and (ct^*) $a \cdot b \leq (a \cdot b) \cdot b$.*

A *residuated weak Heyting algebra* is a BDRG satisfying the conditions (w^*) and (ct^*). Let RWH be the class of all such algebras. A class of algebras is *canonical* if it is closed under canonical extensions (see e.g. [10]). We can prove that WH is canonical. In the canonical extension of a WHA, we define a product $\cdot$ and $\leftarrow$ to get a RWHA.

Theorem 1. *For every $\mathcal{L}_S$-sequent $\Gamma \vdash \phi$, $\Gamma \vdash_{\mathsf{wK}_\sigma} \phi$ iff $\mathsf{RWH} \models \Gamma \vdash \phi$.*

For introducing Gentzen-style sequent calculus for RWH, we allow two structure operators $\owedge$ and $\odot$ for $\wedge$ and $\cdot$ respectively. The sequent calculus $\mathsf{G}_{\mathsf{RWH}}$ consists of the following axioms and rules:

$$(\mathrm{Id})\ \phi \vdash \phi, \quad (\top)\ \Gamma \vdash \top, \quad (\bot)\ \Gamma[\bot] \vdash \phi,$$

$$(\to \mathrm{L})\ \frac{\Delta \vdash \phi \quad \Gamma[\psi] \vdash \gamma}{\Gamma[\Delta \odot (\phi \to \psi)] \vdash \gamma}, \quad (\to \mathrm{R})\ \frac{\phi \odot \Gamma \vdash \psi}{\Gamma \vdash \phi \to \psi}, \quad (\leftarrow \mathrm{L})\ \frac{\Gamma[\phi] \vdash \gamma \quad \Delta \vdash \psi}{\Gamma[(\phi \leftarrow \psi) \odot \Delta] \vdash \gamma},$$

$$(\leftarrow \mathrm{R})\ \frac{\Gamma \odot \psi \vdash \phi}{\Gamma \vdash \phi \leftarrow \psi}, \quad (\cdot \mathrm{L})\ \frac{\Gamma[\phi \odot \psi] \vdash \gamma}{\Gamma[\phi \cdot \psi] \vdash \gamma}, \quad (\cdot \mathrm{R})\ \frac{\Gamma \vdash \phi \quad \Delta \vdash \psi}{\Gamma \odot \Delta \vdash \phi \cdot \psi}, \quad (\wedge \mathrm{L})\ \frac{\Gamma[\phi \owedge \psi] \vdash \gamma}{\Gamma[\phi \wedge \psi] \vdash \gamma},$$

$$(\wedge \mathrm{R})\ \frac{\Gamma \vdash \phi \quad \Delta \vdash \psi}{\Gamma \owedge \Delta \vdash \phi \wedge \psi}, \quad (\vee \mathrm{L})\ \frac{\Gamma[\phi] \vdash \gamma, \quad \Gamma[\psi] \vdash \gamma}{\Gamma[\phi \vee \psi] \vdash \gamma}, \quad (\vee \mathrm{R})\ \frac{\Gamma \vdash \phi_i}{\Gamma \vdash \phi_1 \vee \phi_2},$$

$$(\owedge \mathrm{C})\ \frac{\Gamma[\Delta \owedge \Delta] \vdash \phi}{\Gamma[\Delta] \vdash \phi}, \quad (\owedge \mathrm{W})\ \frac{\Gamma[\Delta] \vdash \phi}{\Gamma[\Sigma \owedge \Delta] \vdash \phi}, \quad (\owedge \mathrm{E})\ \frac{\Gamma[\Delta \owedge \Lambda] \vdash \phi}{\Gamma[\Lambda \owedge \Delta] \vdash \phi},$$

$$(\owedge \mathrm{As})\ \frac{\Gamma[(\Delta_1 \owedge \Delta_2) \owedge \Delta_3] \vdash \phi}{\Gamma[\Delta_1 \owedge (\Delta_2 \owedge \Delta_3)] \vdash \phi}, \quad (\odot w^*)\ \frac{\Gamma[\Delta] \Rightarrow \phi}{\Gamma[\Delta \odot \Delta'] \Rightarrow \phi}, \quad (\odot ct^*)\ \frac{\Gamma[(\Lambda \odot \Delta) \odot \Delta] \Rightarrow \phi}{\Gamma[\Lambda \odot \Delta] \Rightarrow \phi}$$

In the rule $(\vee\mathrm{R})$, i is equal to 1 or 2.

Theorem 2. *The following mix rule*

$$(\mathrm{Mix})\ \frac{\Delta \vdash \phi \quad \Gamma[\phi] \ldots [\phi] \vdash \psi}{\Gamma[\Delta] \ldots [\Delta] \vdash \psi}$$

is admissible in $\mathsf{G}_{\mathsf{RWH}}$. Moreover, $\mathsf{G}_{\mathsf{RWH}}$ has the subformula property.

The approach can be extended to cover many extensions of wK_σ. Firstly, by applying the algorithm ALBA [2], one can define inductive sequents in the language $\mathcal{L}_S$ which have first-order correspondents and are canonical.

Consider the set of sequents $\mathcal{L}^\bullet = \{\phi \vdash \psi \mid \phi, \psi$ are terms built from $\top$, $\bot$ and propositional variables using only $\cdot\}$. Given a sequent $(\sigma)\ \chi \vdash \delta \in \mathcal{L}^\bullet$ the propositional variables occurred in which are among $p_1, \ldots, p_n$, the structural rule corresponding to σ is defined as

$$\frac{\delta[\Gamma_1/p_1, \ldots, \Gamma_n/p_n] \Rightarrow \Delta}{\chi[\Gamma_1/p_1, \ldots, \Gamma_n/p_n] \Rightarrow \Delta}(\odot\sigma)$$

where $\delta[\Gamma_1/p_1, \ldots, \Gamma_n/p_n]$ and $\chi[\Gamma_1/p_1, \ldots, \Gamma_n/p_n]$ are obtained from δ and χ by substituting uniformly Γ_i for p_i.

Theorem 3. *Assume that Φ is a set of inductive sequents in $\mathcal{L}$, and $\Psi = \{t \in \mathcal{L}^\bullet \mid s$ corresponds to t for some $s \in \Phi\}$. Then the algebraic sequent $\mathsf{DFNL}^+(\Psi)$ is a conservative extension of $\mathsf{S}_{\mathsf{BDLI}}(\Phi)$, where $\mathsf{S}_{\mathsf{BDLI}}$ is an algebraic sequent system for algebras obtained from WHA by deleting the conditions (C3) and (C4).*

Theorem 4. *For any set of sequents $\Psi \subseteq \mathcal{L}^\bullet$, the (Mix) rule is admissible in the Gentzen-style sequent system $\mathsf{G}_{\mathsf{DFNL}^+}(\odot\Psi)$, where $\odot\Psi = \{\odot\sigma \mid \sigma \in \Psi\}$.*

Theorem 5. *For any set of sequents $\Psi \subseteq \mathcal{L}^\bullet$, (1) $\Gamma \vdash_{\mathsf{G}_{\mathsf{DFNL}^+}(\odot\Psi)} \phi$ iff $\mathsf{Alg}^\bullet(\Psi) \models \Gamma \vdash \phi$; (2) if every subformula of δ is a subformula of χ for each sequent $\chi \vdash \delta \in \Psi$, then $\mathsf{G}_{\mathsf{DFNL}^+}(\odot\Psi)$ has the subformula property.*

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References

1. W. Buszkowski. Lambek calculus and substructural logics. *Linguistic Analysis* 36(1): 1–15, 2006.
2. W. Conradie, S. Ghilardi and A. Palmigiano. Unified correspondence. In A. Baltag and S. Smets *Johan van Benthem on Logic and Information Dynamics*, pp.933-976, Springer, 2014.
3. S. Celani and R. Jansana. A closer look at some subintuitionistic logics. *Notre Dame Journal of Formal Logic* 42, 225–255, 2003.
4. H. Wansing. *Displaying as temporalizing*. In S. Akama, editor, *Logic, Language and Computation*, pages 159-178, Kluwer Academic Publishers, 1997.
5. K. Došen. Modal translations in K and D. In M. de Rijke, editor, *Diamonds and Defaults*, pages 103–127. Kluwer Academic Publishers, 1993.
6. S. Celani and R. Jansana. Bounded distributive lattices with strict implication. *Mathematical Logic Quarterly*, 51(3):219–246, 2005.
7. R. Ishigaki and R. Kashima. Sequent calculi for some strict implication logics. *Logic Journal of IGPL*, 16(2):155–174, 2007.
8. G. Corsi. Weak logics with strict implication. *Zeitschrift f. mathematische Logik u. Grundlagen d*, 33:389–406, 1987.
9. A. Visser. A propositional logic with explicit fixed points. *Studia Logica*, 40(2):155–175, 1981.
10. M. Geherk and J. Harding. Bounded lattice expansion. *Journal of Algebra*, 238:345–371, 2001.
11. N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*. Elsevier, 2007.