Representation of Free Finitely Generated Weak Nilpotent Minimum Algebras

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The *Monoidal T-norm based Logic* MTL was introduced in [5], and in [7] it was shown that MTL is the logic of all left-continuous t-norms and their residua. Since MTL can be axiomatized as an extension of the *Full Lambek calculus* ([6]), in the hierarchy of substructural logics, MTL is the weakest many-valued logic to be t-norm based. Indeed, a t-norm admits a residuum if and only if it is left-continuous .

The algebraic counterpart of MTL logic is given by a variety of residuated lattices called *MTL algebras*. Interestingly, the class of unary operation $': [0,1] \to [0,1]$ arising as negation operations of MTL algebras over $[0,1]$ coincides with the class of *weak negation* functions; that is, unary operations over $[0,1]$ such that, for all $a, b \in [0,1]$: $0' = 1$; $a \leq b$ implies $b' \leq a'$; and, $a \leq a''$ ([4]).

Given a weak negation $': [0,1] \to [0,1]$, it is possible to equip $[0,1]$ with an MTL algebraic structure by defining the t-norm operation as follows, for all $a, b \in [0,1]$:

$$a * b = \begin{cases} 0 & \text{if } a \leq b', \\ \min(a,b) & \text{otherwise.} \end{cases} \tag{1}$$

Indeed, the class of all weak negations, intended as the MTL algebraic structures over $[0,1]$ described above, generates a subvariety of MTL algebras called *weak nilpotent minimum* algebras, or *WNM algebras*.

WNM algebras play a prominent role in the study of subvarieties of MTL algebras. Indeed, in their lattice of subvarieties we find the varieties of *Gödel algebras* and *Nilpotent Minimum algebras*, that is prelinear *Heyting algebras* and prelinear *Nelson algebras* ([3]) respectively. Moreover, classes of algebras related to the fundamental *drastic product* t-norm [3], that is *RDP algebras* ([9]) and *DP algebras* [1], are both subvarieties of WNM algebras.

[3] As the smallest possible t-norm, the drastic product is considered one of the fundamental t-norms, but there is no a many-valued logic based on it, since it is not left-continuous.

We will present a concrete, combinatorial representation of free finitely generated algebras in the variety of WNM algebras. The problem is interesting for both logical and algebraic reasons.

On the logical side, we will explicitly construct elements of the free finitely generated WNM algebras that are exactly the truth-functions of WNM logic. This construction allows further investigations of the deductive system, such as unification and interpolation properties.

On the algebraic side, the problem is nontrivial because it requires a description of finitely generated WNM chains, nice enough to study a certain subalgebra of their direct product. Exploiting the fact that the variety of WNM algebras is *locally finite*, a combinatorial description of WNM chains is reachable, in sharp contrast with MTL algebras, where a nice description of chains is unknown (and hard).

The presented approach has been already used to study special cases of WNM algebras, the aforementioned Gödel algebras and NM algebras in [2], and RDP algebras in [8]. We will generalize such results to the entire class of WNM algebras.

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