The Canonical FEP Construction for Residuated Lattice Ordered Algebras

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A class of algebras has the *finite embeddability property (FEP, for short)* if every finite partial subalgebra of a member of the class can be embedded into a *finite* member of the class. If a class of algebras has the FEP then it is generated as a quasivariety by its finite members hence, if it is finitely axiomatized it has a decidable quasi-equational theory (and also a decidable universal theory) [1]. If a class of algebras with the FEP algebraizes a logic, then the logic has the finite model property; in fact, it has the 'strong finite model property' meaning that if a rule is refutable in the logic, then it is refutable in a finite model of the logic. Consequently, decidability results for a logic may also be obtained by proving the FEP for its algebraic model class.

In this talk we consider 'residuated lattice ordered algebras', which are algebras with an underlying lattice order and with unary and binary operations that are 'residuated'. A unary function $f : A \to A$ is called *residuated* if there exists a function $g : A \to A$ such that $f(a) \leq b$ if, and only if, $a \leq g(b)$ for all $a, b \in A$. A binary function $\circ : A \times A \to A$ is called *residuated* if there exist functions $\backslash, / : A \times A \to A$ such that $a \circ b \leq c$ if, and only if, $b \leq a \backslash c$ if, and only if, $a \leq c/b$ for all $a, b, c \in A$. The operations g and $\backslash, /$ are called the *residuals* of f and $\circ$, respectively. The language of the algebras then consists of the lattice operations, a finite set of residuated unary and binary operations and all their residuals, and possibly a finite set of constants. Such classes of residuated lattice ordered algebras are often the algebraic models of substructural logics, in which the residuals $\backslash, /$ serve as implication operations (see, e.g., [3]). In the particular case that $\circ$ is commutative, its two residuals coincide and are usually denoted by $\to$. By a *residuated lattice* is usually meant a residuated lattice ordered algebra with single binary residuated operation.

The FEP was obtained in [1] for the class of residuated lattices whose residuated operation is commutative and has an identity element which is also the greatest element of the lattice (i.e., the residuated lattice is *integral*). The method was extended to various classes of residuated (lattice) ordered algebras in [10], where it was shown that the construction is based on the MacNeille completion in the sense that, under certain conditions on the parameters, the construction produces the MacNeille completion of the algebra under consideration. A natural question that arises is whether or not it would be possible to use a different lattice completion method to obtain a different proof of the FEP.

In this talk we present a new FEP construction for (decreasing) residuated lattice ordered algebras that is based on the *canonical extension*, again in the sense that, under certain conditions on the parameters, the construction produces the canonical extension of the algebra under consideration. The 'decreasing' condition is a generalization of integrality; it means that each residuated operation satisfies $f(x) \leq x$ if unary, and $x \circ y \leq x \wedge y$ if binary.

The canonical extension was first introduced for Boolean algebras with operators [7] and has since been studied in various other contexts (see, e.g., [6, 4, 2, 5, 8]). Abstractly, the canonical extension of a (bounded) lattice is characterized as the unique completion of the lattice that is compact and dense. A concrete construction of the canonical extension of a lattice was presented in [4] (first appearing in [9]). This construction makes use of the Galois connection formed between the polarities of a relation between sets of ideals and filters of the lattice.

Starting with a residuated lattice ordered algebra and a finite partial subalgebra thereof, the 'canonical FEP construction' presented here makes use of a Galois connection between the polarities of a relation between restricted sets of ideals and filters to obtain a lattice. On this lattice, the operations of the partial algebra are then extended to full operations. Traditionally there are two standard extensions of operations defined for the canonical extensions of lattices, known as the σ- and π-extensions. As we show, the extensions we use are alternate descriptions of the σ- and π-extensions that do not rely on the notions of closed or open elements. The algebra obtained in this way is shown to be a residuated lattice ordered algebra; moreover, it is finite if the original algebra is decreasing.

In addition, we show that the canonical FEP construction produces an algebra whose underlying lattice satisfies a weak form of compactness, called *internal compactness* [4]. This property is not generally obtained for the construction based on the MacNeille completion, and we show, by example, that this construction generally produces a different algebra to that obtained by the construction in [1, 10].

Finally we discuss preservation of properties through the canonical FEP construction. That is, we provide syntactical descriptions of equalities and inequalities that hold in the constructed algebra if they hold in the original algebra. Such preservations make use of approximation terms based on the σ- and π-extensions, again using the alternate description of these. These results are reminscent of the Sahlqvist canonicity results.

References

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