Trakhtenbrot theorem and first-order axiomatic extensions of MTL

Matteo Bianchi[1] and Franco Montagna[2]

[1] Department of Computer Science, Università degli Studi di Milano, Via Comelico 39/41, 20135, Milano, Italy
matteo.bianchi@unimi.it

[2] Department of Information Engineering and Mathematics, Università degli Studi di Siena, via Roma 56, 53100, Siena, Italy

Abstract. In 1950, B.A. Trakhtenbrot showed that the set of first-order tautologies associated to finite models is not recursively enumerable. In 1999, P. Hájek generalized this result to the first-order versions of Łukasiewicz, Gödel and Product logics, w.r.t. their standard algebras. In this talk we extend the analysis to the first-order versions of axiomatic extensions of MTL. Our main result is the following. Let $\mathbb{K}$ be a class of non-trivial MTL-chains: then the set of all first-order tautologies associated to the finite models over chains in $\mathbb{K}$, $\mathrm{fTAUT}_{\forall}^{\mathbb{K}}$, is Π_1^0-hard. Let $\mathrm{TAUT}_{\mathbb{K}}$ be the set of propositional tautologies of $\mathbb{K}$: if $\mathrm{TAUT}_{\mathbb{K}}$ is decidable, we have that $\mathrm{fTAUT}_{\forall}^{\mathbb{K}}$ is in Π_1^0. We have similar results also if we expand the language with the Δ operator.

Extended abstract

In [6], B.A. Trakhtenbrot showed that the set of first-order tautologies associated to finite models is not recursively enumerable, in classical first-order logic. In particular, it is known that such set is Π_1^0-complete: in [8,2] it is shown that the theorem works also with languages containing only predicates, with at least a binary one, and without equality. This result implies the fact that the completeness w.r.t. finite models does not hold, in first-order logic: indeed, the set of theorems of classical predicate logic is Σ_1^0-complete.

One can ask if a similar result holds also in non-classical logics, for example many-valued logics. A first answer was given in [5] by P. Hájek, who generalized Trakhtenbrot theorem to the first-order versions of Łukasiewicz, Gödel and Product logics, with respect to their standard algebras.

In this talk we outline the results of [1], where a generalized version of Trakhtenbrot theorem is presented, for the (first-versions) of the axiomatic extensions of MTL ([4,3]).

The main results that we will discuss are the following ones:

- Let $\mathbb{K}$ be a class of non-trivial MTL-chains: then the set of all first-order tautologies associated to the finite models over chains in $\mathbb{K}$, $\mathrm{fTAUT}_{\forall}^{\mathbb{K}}$, is Π_1^0-hard. Let now $\mathrm{TAUT}_{\mathbb{K}}$ be the set of propositional tautologies of $\mathbb{K}$: if

$\mathrm{TAUT}_{\mathbb{K}}$ is decidable, we have that $\mathrm{fTAUT}_{\forall}^{\mathbb{K}}$ is in Π_1^0. As a consequence, if L is a consistent axiomatic extension of MTL, and $\mathbb{K}$ is the class of all L-chains, then $\mathrm{fTAUT}(\mathrm{L}\forall) \overset{\mathrm{def}}{=} \mathrm{fTAUT}_{\forall}^{\mathbb{K}}$ is Π_1^0-hard: moreover, if L is decidable, then $\mathrm{fTAUT}(\mathrm{L}\forall)$ is Π_1^0-complete.
- By the previous results we have that the decidability of a consistent axiomatic extension L of MTL is a sufficient condition for the Π_1^0-completess of $\mathrm{fTAUT}(\mathrm{L}\forall)$. Is it also necessary? We will show that the answer is positive if L is recursively axiomatizable: however, we have negative results if we expand the language of L with constants, and L is not recursively axiomatizable.
- We conclude by showing some negative results about the expansions of MTL with the Δ operator, and discussing some open problems.

References

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