States on finite GBL*-algebras

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GBL-algebras are the divisible residuated lattices, and the variety of commutative bounded GBL-algebras has both the variety of MV-algebras and the variety of Heyting algebras (hence the one of Gödel algebras) as subvarieties. Finite GBL-algebras are always bounded and commutative [6], and they can be represented as weighted posets [7].

In [3] we introduced *GBL-pairs*. A GBL-pair is a pair (H, G), where H is a Heyting algebra and G is a suitable subgroup of its automorphism group, that induces an equivalence relation $\sim$ on H. GBL-pairs generalize, for the finite case, similar structures introduced in [5], [2], [11] and [12]. For every finite GBL-pair (H, G) the set H/G of equivalence classes can be endowed with a structure of GBL-algebra, and all finite GBL-algebras arise in this way.

The representation of GBL-algebras as GBL-pairs is extended to a categorical equivalence in [4], whence it comes natural to make explicit a further operation $\oplus$, which coincides with the monoidal sum in the particular case of MV-algebras. Thus we define a finite GBL*-algebra as a structure $\langle X, \odot, \oplus, \to, \top, \bot\rangle$ such that X is a finite set, $\langle X, \odot, \top\rangle$ and $\langle X, \oplus, \bot\rangle$ are commutative monoids and, for all $a, b, c \in X$:

- $a \to (a \oplus b) = \top$;
- $a \odot (a \to b) = b \odot (b \to a)$;
- $(a \odot b) \to c = b \to (a \to c)$;
- $(a \oplus b) \to b = a \to (a \odot b)$.

Note that, in the subclass of Heyting algebras (and, in particular, in the one of Gödel algebras), $\odot$ and $\oplus$ coincide, respectively, with $\wedge$ and $\vee$, and in every finite GBL*-algebra $\wedge$ and $\vee$ are derived operations.

In this talk we introduce a notion of *state* on GBL*-algebras. In order to generalize established notions of state on MV-algebras [8] and Gödel algebras [1], we require that a state $s : X \to [0, 1]$ on a GBL*-algebra X satisfy the following conditions, for all $a, b \in X$:

- $s(\bot) = 0$, $s(\top) = 1$;
- if $a \leq b$, then $s(a) \leq s(b)$;
- $s(a \odot b) + s(a \oplus b) = s(a) + s(b)$.

As a particular case we have states on Heyting algebras (and, in particular, on Gödel algebras), that are valuations (in the sense of [10]) on their distributive lattice reduct.

Given a GBL-pair (H, G), we say that a state $v : H \to [0, 1]$ is invariant with respect to the equivalence relation if $a \sim b$ implies $v(a) = v(b)$. We show a bijective correspondence between invariant states on H and states on H/G, providing a further interpretation of states on GBL*-algebras. The correspondence is given in the following way: for every $x \in H$ we set $s([x]) = v(x)$, where $[x]$ is the class of x in H/G.

Using *Möbius inversion formula* [9], we show that states on a GBL*-algebra X correspond to density functions on the weighted poset that represents X, and to density functions on the poset of join-prime elements of the Heyting algebra H in the GBL-pair that represents X.

Finally, let X be a GBL*-algebra and $s : X \to [0, 1]$ be a state such that $s(x) > 0$ for all $x > \bot$. We introduce the notion of *conditional state* $s(x|i)$, where $i > \bot$ is an idempotent. We obtain the following equation, analogous to the *law of total probability*:

$$s(x) = \sum_j e(j)s(x|j)s(j), \tag{1}$$

where j varies over join-prime idempotents and e is a function, taking values in $\mathbb{Z}$, defined through the *Möbius function* [9]. In the particular case of MV-algebras, $e(j) = 1$ for all j join-prime idempotents.

In Equation (1) the function $s(\cdot|j)$, that is a state on X, can be thought of as a *truth valuation*, since it gives, when X is a MV-algebras or a Gödel algebra, all the homomorphisms from X to the algebra $[0, 1]$ as $s(\cdot)$ varies over the states and j over the join-prime idempotents. The value $s(x|j)$ can also be interpreted as a *degree of truth* of x to belong to the prime filter bounded by j.

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