On everywhere strongly logifiable algebras

Tommaso Moraschini

Department of logic, history and philosophy of science, University of Barcelona (UB), Montalegre 6, E-08001 Barcelona, Spain
`tommaso.moraschini@ub.edu`

In the late 80's Blok and Pigozzi provided a uniform framework for the algebraic approach to the analysis of propositional logics, namely the theory of algebraizability [1]. In general the equivalent algebraic semantics $\mathsf{Alg}^*\mathcal{L}$ of an algebraizable logic $\mathcal{L}$ is a generalized quasi-variety. However, most of the well-known algebraizable logics have an equivalent algebraic semantics that is a variety. This posed the natural question, sometimes called in the literature "variety problem", of explaining this phenomenon by finding some meaningful sufficient conditions under which the equivalent algebraic semantics of an algebraizable logic is a variety: conditions of this kind have been obtained for example in [2,3,6,7,8]. Following [1] a logic $\mathcal{L}$ is called ***strongly algebraizable*** if it is algebraizable and $\mathsf{Alg}^*\mathcal{L}$ is a variety.

The starting point of this talk is the attempt to define what does it mean that a finite (non-trivial) algebra $\boldsymbol{A}$ behaves in the best possible way from the point of view of Blok and Pigozzi's algebraizability theory. Obviously, several definitions can be proposed to formalize this non-mathematical concept. Our idea is to focus on algebras for which some sort of cut-and-paste process, where the "cut" part consists in cutting arbitrary subsets of the universe of the algebras and the "paste" one to pasting them into matrices, always yields a strongly algebraizable logic. More precisely, we say that a finite non-trivial algebra $\boldsymbol{A}$ is ***everywhere strongly logifiable*** when the logic determined by the matrix $\langle \boldsymbol{A}, F\rangle$ is strongly algebraizable with equivalent algebraic semantics $\mathbb{V}(\boldsymbol{A})$, for every $F \in \mathcal{P}(A) \smallsetminus \{\emptyset, A\}$. Primal algebras are natural examples of everywhere strongly logifiable algebras. Nevertheless, these two concepts do not coincide: we will see that for every $n \geq 3$ there is an everywhere strongly logifiable algebra of n elements, which is not primal.

At first sight the notion of an everywhere strongly logifiable algebra seems to belong to the field of algebraic logic rather than to the one of universal algebra. Therefore it is natural to ask whether it is possible to characterize everywhere strongly logifiable algebras by means of familiar algebraic conditions. Luckily it turns out that this in indeed the case. An algebra $\boldsymbol{A}$ is ***constantive*** if for every $a \in A$, there is an (at most unary) term $c_a(x)$ which on $\boldsymbol{A}$ represents the map constantly equal to a. Then we will prove the following:

Theorem 1. *A finite non-trivial algebra $\boldsymbol{A}$ is everywhere strongly logifiable if and only if it is simple, constantive and generates a congruence distributive and n-permutable variety for some $n \geq 2$.*

This non-trivial result somehow confirms the intuition that everywhere strongly logifiable algebras are reasonably close to the primal ones, in the sense that at least in congruence permutable varieties these two concepts coincide. Drawing consequences from this result we obtain a solution to the variety problem for logics determined by matrices of the form $\langle \boldsymbol{A}, \{a\}\rangle$, where $\boldsymbol{A}$ is a finite, non-trivial and constantive algebra. The tools used in the proof of the main theorem are the theory of algebraizable logics and some very basic commutator theory [4] and tame congruence theory [5].

References

1. W. J. Blok and D. Pigozzi. *Algebraizable logics*, volume 396 of *Mem. Amer. Math. Soc.* A.M.S., Providence, January 1989.
2. J. M. Font. On semilattice-based logics with an algebraizable assertional companion. *Reports on Mathematical Logic*, 46:109–132, 2011.
3. J. M. Font and R. Jansana. *A general algebraic semantics for sentential logics*, volume 7 of *Lecture Notes in Logic*. Springer-Verlag, second edition, 2009. Electronic version freely available through Project Euclid at `projecteuclid.org/euclid.lnl/1235416965`.
4. R. Freese and R. McKenzie. *Commutator theory for congruence modular varieties*, volume 125 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1987.
5. D. Hobby and R. McKenzie. *The structure of finite algebras*, volume 76 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1988.
6. R. Jansana. Selfextensional logics with implication. In *Logica universalis*, pages 65–88. Birkhäuser, Basel, 2005.
7. R. Jansana. Self-extensional logics with a conjunction. *Studia Logica*, 84(1):63–104, September 2006.
8. R. Jansana. Algebraizable logics with a strong conjunction and their semi-lattice based companions. *Arch. Math. Logic*, 51:831–861, 2012.