Undecidability in abstract algebraic logic

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Abstract algebraic logic is a theory that provides a general framework for the uniform study of propositional logics. One of its main achievements is the development of the so-called Leibniz and Frege hierarchies (see for example [1,2,3]). In the first one propositional logics are classified according to properties related to the definability of their truth predicates and of logical equivalence. While in the second one they are classified by means of general kinds of replacement properties. In this talk we study the problem of classifying logics in these two hierarchies from a computational point of view; this problem was formulated by Félix Bou in the Seminar on Non-Classical Logics of the University of Barcelona. In particular we will focus on logics defined syntactically by a Hilbert-style calculus and on logics defined semantically by a finite set of finite matrices. Accordingly, the problem of classifying logics in the Leibniz and Frege hierarchies can be formulated both in ***syntactic*** and ***semantic*** terms. Not surprisingly the semantic version of this problem seems easy. In fact it is possible to describe an ***algorithm***, which classifies logics determined by a finite set of finite matrices of finite type into the Leibniz hierarchy. An implementation of it is freely available online [9].

The main goal of this talk will be that of proving that, in the case of syntactically presented logics, the situation is the opposite. More precisely we will show the following:

Theorem 1. *Let K a level of the Leibniz (Frege) hierarchy. The problem of determining whether the logic of a finite Hilbert calculus in a finite language belongs to K is* ***undecidable****.*

Our proof relies on the idea of embedding an undecidable purely algebraic problem into that of classifying logics of Hilbert calculi in the Leibniz and Frege hierarchies. Beginning by the case of the Leibniz hierarchy, the following construction (introduced in [4,5]) will be useful: given a variety of algebras V, the ***basic logic*** $\mathcal{L}_{\mathsf{V}}$ of V is the logic determined by the class of matrices

$$\{\langle \boldsymbol{A}, F \rangle : \boldsymbol{A} \in \mathsf{V} \text{ and } F \subseteq A\}.$$

In general the logic $\mathcal{L}_{\mathsf{V}}$ can fail to be finitely Hilbert-axiomatizable, even if the variety V is finitely based. For example, this is the case for the variety of commutative magmas. Nevertheless we provide an explicit and finite (but not very intuitive) Hilbert calculus for the basic logic $\mathcal{L}_{\mathsf{CR}}$ of the variety CR of commutative rings with unit. This will give us the framework of a finite Hilbert calculus in which we wish to embed ring theoretic undecidable problems.

Accordingly to this intuition, given a Diophantine equation $p \approx 0$, we slightly modify the calculus of $\mathcal{L}_{\mathsf{CR}}$ obtaining a new logic $\mathcal{L}(p)$ in a way such that

$$p \approx 0 \text{ has an integer solution } \iff \mathcal{L}(p) \text{ belongs to } \mathsf{K},$$

where K is an arbitrary level of the Leibniz hierarchy. Since the problem of determining whether a given Diophantine equation has an integer solution is undecidable [6], we conclude that the problem of classifying logics of Hilbert calculi into the Leibniz hierarchy must be undecidable too.

In the case of the Frege hierarchy we will make use of a different, but similar, proof strategy. For every equation $\alpha \approx \beta$ of relational algebras RA in one variable x, we define (trough a finite Hilbert calculus) an algebraizable logic $\mathcal{L}(\alpha, \beta)$ in a way such that

$$\begin{aligned}\mathsf{RA} \vDash \alpha \approx \beta &\iff \mathcal{L}(\alpha, \beta) \text{ is inconsistent} \\ &\iff \mathcal{L}(\alpha, \beta) \text{ belongs to } \mathsf{K},\end{aligned}$$

where K is an arbitrary level of the Frege hierarchy. The fact that only variable x appears in $\alpha \approx \beta$ plays a central role in the proof of the above equivalences. Since the equational theory in one variable x of relational algebras is undecidable [7,8], we conclude that the problem of classifying logics of Hilbert calculi into the Frege hierarchy must be undecidable too.

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