A Hilbert space operator representation of generalized effect algebras of bilinear forms and measures

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Extended abstract

A generalization of an effect algebra which omits the top element is called a *generalized effect algebra*. Many relevant sets of positive linear operators on a complex Hilbert space $\mathcal{H}$ posses the structure of a generalized effect algebra. Basic examples are positive cones in po-groups of linear operators (e.g. the set of all bounded positive operators $\mathcal{B}^+(\mathcal{H})$ in the po-group $\mathcal{B}(\mathcal{H})$). There are also others, such as the set of all positive operators $\mathcal{V}(\mathcal{H})$ ([8]). One of the most important results was a determination of necessary and sufficient conditions for an abstract effect algebra to be representable by linear operators (see [7]). This result was extended to the case of MV-algebras (see [6]), generalized effect algebras (see [5]) and abelian po-groups (see [1]).

In [2] Dvurečenskij and Janda investigated several sets of positive bilinear forms and they have shown that there is an analogy to the linear operator case, i.e. some of them also posses a structure of a generalized effect algebra. We show ([3]) that although the set of all positive symmetric bilinear forms is richer than the set of all symmetric linear operators, there exists an embedding of generalized effect algebras from the set of positive bilinear forms into the set of linear operators on a different Hilbert space (it is representable).

Theorem 1. [3] *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and let be a linear subspace $\overline{D} \in \mathcal{H}$. Then there exists an order determining set M of generalized states on $\mathcal{V}_{fD}(\mathcal{H})$ and an embedding from the generalized effect algebra $(\mathcal{V}_{fD}(\mathcal{H}); +_{\mathcal{V}_{fD}(\mathcal{H}))}, o_b)$ of positive bilinear forms on D into the generalized effect algebra of positive linear operators $(Pos(Symm(M)); +_{|Pos(Symm(M))}, \mathbf{0})$ on the dense subspace $\mathcal{E}_{lin}(M)$ of the Hilbert space $l_2(M)$.*

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In [4], we extended ideas of generalized effect algebras approach to finitely additive Gleason measures, regular measures, and σ-measures.

Theorem 2. [4] *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Let $\mathrm{Reg}_f(\mathcal{H})$ be the set of regular finitely additive measures m on $\mathcal{L}(\mathcal{H})$ with the L-S density property such that if m is $\mathcal{P}_1(\mathcal{H})$-bounded, then $D(m) = \mathcal{H}$. Let us define a partial operation $\oplus$ on $\mathrm{Reg}_f(\mathcal{H})$: For $m_1, m_2 \in \mathrm{Reg}_f(\mathcal{H})$, $m_1 \oplus m_2$ is defined if and only if m_1 or m_2 is $\mathcal{P}_1(\mathcal{H})$-bounded or $D(m_1) = D(m_2)$ and then $m_1 \oplus m_2 := m_1 + m_2$. Then $(\mathrm{Reg}_f(\mathcal{H}); \oplus, o)$ is a generalized effect algebra.*

We can extend results about representability and embed into generalized effect algebra of operators on Hilbert space $\mathcal{H}$ even some generalized effect algebras of regular finitely additive measures, as follows.

Theorem 3. *Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and let $\overline{D} \in \mathcal{H}$ be a linear subspace of $\mathcal{H}$. Let $\mathrm{Reg}_{fD}(\mathcal{H})$ be the set of regular finitely additive measures m on $\mathcal{L}(\mathcal{H})$ with the L-S density property such that if m is $\mathcal{P}_1(\mathcal{H})$-bounded, then $D(m) = \mathcal{H}$, in other case $D(m) = D$. Then $\mathrm{Reg}_{fD}(\mathcal{H})$ forms a sub-generalized effect algebra of $(\mathrm{Reg}_f(\mathcal{H}); \oplus, o)$. Moreover, there exists an order determining set M of generalized states on $\mathrm{Reg}_{fD}(\mathcal{H})$ and an embedding from the generalized effect algebra $(\mathrm{Reg}_{fD}(\mathcal{H}); +_{\mathrm{Reg}_{fD}(\mathcal{H})}), o)$ into the generalized effect algebra of positive linear operators $(Pos(Symm(M)); +_{|Pos(Symm(M))}, \mathbf{0})$ on the dense subspace $\mathcal{E}_{lin}(M)$ of the Hilbert space $l_2(M)$.*

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