Uniform Interpolation and Compact Congruences

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The following remarkable feature of intuitionistic propositional logic (**IPC**) was established by A. M. Pitts in [3]. Given any formula $\alpha(\bar{x}, \bar{y})$ (using brackets as usual to indicate variables that may occur in the formula), there exist formulas $\alpha^L(\bar{y})$ and $\alpha^R(\bar{y})$, *left* and *right uniform interpolants* of α, respectively, such that for any formula $\beta(\bar{y}, \bar{z})$,

$$\beta \vdash_{\mathbf{IPC}} \alpha \quad \Leftrightarrow \quad \beta \vdash_{\mathbf{IPC}} \alpha^L \qquad \text{and} \qquad \alpha \vdash_{\mathbf{IPC}} \beta \quad \Leftrightarrow \quad \alpha^R \vdash_{\mathbf{IPC}} \beta.$$

All seven intermediate logics admitting Craig interpolation also admit uniform interpolation; however, although the modal logic **K** admits both properties, its extension **S4** admits only Craig interpolation and not uniform interpolation (see [1] for details and references).
Uniform interpolation for a logic may be viewed as a weaker form of quantifier elimination. This idea is exploited in the monograph [1] of Ghilardi and Zawadowski to show that under certain conditions, satisfied in particular by **IPC** and **K**, uniform interpolation for a logic implies the existence of a model completion for a corresponding variety (equational class) of algebras.
In this work, we investigate uniform interpolation in a universal algebraic setting. Following the category-theoretic work in [1], we obtain algebraic characterizations of the property of existence of left and right uniform interpolants. Moreover, we identify, among varieties of algebras corresponding to substructural and many-valued logics, several varieties that admit and do not admit these properties.

In the remainder of this abstract, we give a more technical description of our main results. Let us fix an algebraic language $\mathcal{L}$ and a variety $\mathcal{V}$ of $\mathcal{L}$-algebras. We denote by $\mathbf{F}_{\mathcal{V}}(\overline{x})$ the free $\mathcal{V}$-algebra over a set of variables $\overline{x}$. The *deductive interpolation property* [2] for $\mathcal{V}$ is easily shown to be equivalent to: for any set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a set of equations $\Pi(\overline{y})$ such that for any equation $\varepsilon(\overline{y}, \overline{z})$, $\Sigma \models_{\mathcal{V}} \varepsilon$ iff $\Pi \models_{\mathcal{V}} \varepsilon$. We now formulate a uniform version of this property: $\mathcal{V}$ has *right uniform deductive interpolation* if, for any finite $\overline{x}$, $\overline{y}$ and any *finite* set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a *finite* set of equations $\Pi(\overline{y})$ such that for any equation $\varepsilon(\overline{y}, \overline{z})$, $\Sigma \models_{\mathcal{V}} \varepsilon$ iff $\Pi \models_{\mathcal{V}} \varepsilon$.

In Theorem 1 below, we translate the above definition into a property of free finitely generated algebras of $\mathcal{V}$. To this end, note first that any homomorphism

$f\colon \mathbf{A} \to \mathbf{B}$ lifts to an adjunction $f^*\colon \mathrm{Con}(\mathbf{A}) \leftrightarrows \mathrm{Con}(\mathbf{B})\colon f^{-1}$ between the congruence lattices, with f^* (direct image) left adjoint to f^{-1} (inverse image). Moreover, the map f^* restricts correctly to the sub-join-semilattices of compact (i.e., finitely generated) congruences, $\mathrm{KCon}(\mathbf{A})$ and $\mathrm{KCon}(\mathbf{B})$. We call the restriction of f^* to compact congruences the *compact lift of* f. By general lattice-theoretic considerations, the compact lift $f^*\colon \mathrm{KCon}(\mathbf{A}) \to \mathrm{KCon}(\mathbf{B})$ has a right adjoint if, and only if, f^{-1} preserves compact congruences. In this case, the restriction of f^{-1} to compact congruences is that right adjoint. As a first characterization of right uniform deductive interpolation, we have the following.

Theorem 1. *For any variety $\mathcal{V}$, the following are equivalent:*

1. *$\mathcal{V}$ has right uniform deductive interpolation;*
2. *(a) for any finite $\overline{x}$, $\overline{y}$, the compact lift of $\mathbf{F}_{\mathcal{V}}(\overline{x}) \hookrightarrow \mathbf{F}_{\mathcal{V}}(\overline{x}, \overline{y})$ has a right adjoint,*
   and
   (b) $\mathcal{V}$ has deductive interpolation.
3. *for any $\overline{x}$, $\overline{y}$, the compact lift of $\mathbf{F}_{\mathcal{V}}(\overline{x}) \hookrightarrow \mathbf{F}_{\mathcal{V}}(\overline{x}, \overline{y})$ has a right adjoint.*

Examples. Heyting algebras have right uniform deductive interpolation by [3] and the fact that any Heyting algebra $\mathbf{A}$ is dually isomorphic to $\mathrm{KCon}(\mathbf{A})$. Note that (2a) in Theorem 1 is automatically true in any variety for which any congruence on a finitely generated free algebra is compact. In particular, any locally finite variety $\mathcal{V}$ with deductive interpolation has right uniform deductive interpolation. Moreover, abelian groups, abelian ℓ-groups and MV-algebras all have right uniform deductive interpolation. On the other hand, in the variety of algebras for the modal logic S4, (2a) does not hold [1], and (2a) also fails in the variety of groups.[4]

In the next theorem, we show that property (2a) in Theorem 1 guarantees the existence of right adjoints for compact lifts of arbitrary homomorphisms between finitely presented algebras.

Theorem 2. *For any variety $\mathcal{V}$, the following are equivalent:*

1. *for any finite $\overline{x}$, $\overline{y}$, the compact lift of $\mathbf{F}_{\mathcal{V}}(\overline{x}) \hookrightarrow \mathbf{F}_{\mathcal{V}}(\overline{x}, \overline{y})$ has a right adjoint;*
2. *for any homomorphism $f\colon \mathbf{A} \to \mathbf{B}$ between finitely presented algebras of $\mathcal{V}$, the compact lift of f has a right adjoint.*

To prove this theorem, we show that one may choose appropriate presentations of $\mathbf{A}$ and $\mathbf{B}$ so that the right adjoint for the compact lift of f can be constructed from the right adjoints that are assumed to exist in (1).

We say that $\mathcal{V}$ has *left uniform deductive interpolation* if, for any finite set of equations $\Delta(\overline{y}, \overline{z})$, there exists a finite set of equations $\Pi(\overline{y})$ such that for any set of equations $\Sigma(\overline{x}, \overline{y})$, $\Sigma \models_{\mathcal{V}} \Delta$ iff $\Sigma \models_{\mathcal{V}} \Pi$. Theorem 1 holds if one replaces

[4] M. Sapir, personal communication

'right' by 'left' throughout. However, the property of left uniform interpolation is not entirely analogous to that of right uniform interpolation.

Examples. As above, Heyting algebras have left uniform deductive interpolation by [3]. It follows from the 'left' version of Theorem 1 that a locally finite variety $\mathcal{V}$ has left uniform deductive interpolation if, and only if, $\mathcal{V}$ has deductive interpolation *and* the compact lift of $\mathbf{F}_{\mathcal{V}}(\overline{x}) \hookrightarrow \mathbf{F}_{\mathcal{V}}(\overline{x}, \overline{y})$ preserves intersections. In particular, we use these observations to give an algebraic proof that the variety of Brouwerian meet-semilattices does not have left uniform deductive interpolation.

One may now naturally wonder if an analogous result to Theorem 2 holds for left adjoints. It turns out that an additional condition is needed. We call a join-semilattice *dually Brouwerian* if the operation of binary join has a left residual.

Theorem 3. *For any variety $\mathcal{V}$, the following are equivalent:*

1. *for any finite $\overline{x}$, $\overline{y}$, the compact lift of $\mathbf{F}_{\mathcal{V}}(\overline{x}) \hookrightarrow \mathbf{F}_{\mathcal{V}}(\overline{x}, \overline{y})$ has a left adjoint, and $\mathrm{KCon}(\mathbf{F}_{\mathcal{V}}(\overline{x}))$ is a dually Brouwerian join-semilattice;*
2. *for any homomorphism $f \colon \mathbf{A} \to \mathbf{B}$ between finitely presented algebras of $\mathcal{V}$, the compact lift of f has a left adjoint.*

For the proof of this theorem, we first observe that, for any algebra $\mathbf{A}$, the join-semilattice $\mathrm{KCon}(\mathbf{A})$ is dually Brouwerian if, and only if, the compact lift of any surjective homomorphism $p \colon \mathbf{A} \twoheadrightarrow \mathbf{B}$ has a left adjoint. This characterization is subsequently combined with an argument similar to that in the proof of Theorem 2.

References

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