Uniform Interpolation and Compact Congruences

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The following remarkable feature of intuitionistic propositional logic (\textbf{IPC}) was established by A. M. Pitts in [3]. Given any formula $\alpha(\overline{x}, \overline{y})$ (using brackets as usual to indicate variables that may occur in the formula), there exist formulas $\alpha^L(\overline{y})$ and $\alpha^R(\overline{y})$, left and right uniform interpolants of $\alpha$, respectively, such that for any formula $\beta(\overline{y}, \overline{z})$,

$$\beta \vdash_{\text{IPC}} \alpha \iff \beta \vdash_{\text{IPC}} \alpha^L \quad \text{and} \quad \alpha \vdash_{\text{IPC}} \beta \iff \alpha^R \vdash_{\text{IPC}} \beta.$$ 

All seven intermediate logics admitting Craig interpolation also admit uniform interpolation; however, although the modal logic $\textbf{K}$ admits both properties, its extension $\textbf{S4}$ admits only Craig interpolation and not uniform interpolation (see [1] for details and references).

Uniform interpolation for a logic may be viewed as a weaker form of quantifier elimination. This idea is exploited in the monograph [1] of Ghilardi and Zawadowski to show that under certain conditions, satisfied in particular by $\textbf{IPC}$ and $\textbf{K}$, uniform interpolation for a logic implies the existence of a model completion for a corresponding variety (equational class) of algebras.

In this work, we investigate uniform interpolation in a universal algebraic setting. Following the category-theoretic work in [1], we obtain algebraic characterizations of the property of existence of left and right uniform interpolants. Moreover, we identify, among varieties of algebras corresponding to substructural and many-valued logics, several varieties that admit and do not admit these properties.

In the remainder of this abstract, we give a more technical description of our main results. Let us fix an algebraic language $\mathcal{L}$ and a variety $\mathcal{V}$ of $\mathcal{L}$-algebras. We denote by $\mathbf{F}_\mathcal{V}(\overline{x})$ the free $\mathcal{V}$-algebra over a set of variables $\overline{x}$. The deductive interpolation property [2] for $\mathcal{V}$ is easily shown to be equivalent to: for any set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a set of equations $\Pi(\overline{y})$ such that for any equation $\varepsilon(\overline{y}, \overline{z})$, $\Sigma \models_{\mathcal{V}} \varepsilon$ iff $\Pi \models_{\mathcal{V}} \varepsilon$. We now formulate a uniform version of this property: $\mathcal{V}$ has right uniform deductive interpolation if, for any finite $\overline{x}$, $\overline{y}$ and any finite set of equations $\Sigma(\overline{x}, \overline{y})$, there exists a finite set of equations $\Pi(\overline{y})$ such that for any equation $\varepsilon(\overline{y}, \overline{z})$, $\Sigma \models_{\mathcal{V}} \varepsilon$ iff $\Pi \models_{\mathcal{V}} \varepsilon$.

In Theorem 1 below, we translate the above definition into a property of free finitely generated algebras of $\mathcal{V}$. To this end, note first that any homomorphism
$f: A \to B$ lifts to an adjunction $f^*: \text{Con}(A) \simeq \text{Con}(B): f^{-1}$ between the congruence lattices, with $f^*$ (direct image) left adjoint to $f^{-1}$ (inverse image). Moreover, the map $f^*$ restricts correctly to the sub-join-semilattices of compact (i.e., finitely generated) congruences, $\text{KCon}(A)$ and $\text{KCon}(B)$. We call the restriction of $f^*$ to compact congruences the \textit{compact lift of} $f$. By general lattice-theoretic considerations, the compact lift $f^*: \text{KCon}(A) \to \text{KCon}(B)$ has a right adjoint if, and only if, $f^{-1}$ preserves compact congruences. In this case, the restriction of $f^{-1}$ to compact congruences is that right adjoint. As a first characterization of right uniform deductive interpolation, we have the following.

**Theorem 1.** For any variety $\mathcal{V}$, the following are equivalent:

1. $\mathcal{V}$ has right uniform deductive interpolation;
2. (a) for any finite $\bar{x}, \bar{y}$, the compact lift of $F\mathcal{V}(\bar{x}) \hookrightarrow F\mathcal{V}(\bar{x}, \bar{y})$ has a right adjoint, and
   
   (b) $\mathcal{V}$ has deductive interpolation.
3. for any $\bar{x}, \bar{y}$, the compact lift of $F\mathcal{V}(\bar{x}) \hookrightarrow F\mathcal{V}(\bar{x}, \bar{y})$ has a right adjoint.

**Examples.** Heyting algebras have right uniform deductive interpolation by \cite{3} and the fact that any Heyting algebra $A$ is dually isomorphic to $\text{KCon}(A)$. Note that (2a) in Theorem 1 is automatically true in any variety for which any congruence on a finitely generated free algebra is compact. In particular, any locally finite variety $\mathcal{V}$ with deductive interpolation has right uniform deductive interpolation. Moreover, abelian groups, abelian $\ell$-groups and MV-algebras all have right uniform deductive interpolation. On the other hand, in the variety of algebras for the modal logic $S4$, (2a) does not hold \cite{1}, and (2a) also fails in the variety of groups.\footnote{M. Sapir, personal communication}

In the next theorem, we show that property (2a) in Theorem 1 guarantees the existence of right adjoints for compact lifts of arbitrary homomorphisms between finitely presented algebras.

**Theorem 2.** For any variety $\mathcal{V}$, the following are equivalent:

1. for any finite $\bar{x}, \bar{y}$, the compact lift of $F\mathcal{V}(\bar{x}) \hookrightarrow F\mathcal{V}(\bar{x}, \bar{y})$ has a right adjoint;
2. for any homomorphism $f: A \to B$ between finitely presented algebras of $\mathcal{V}$, the compact lift of $f$ has a right adjoint.

To prove this theorem, we show that one may choose appropriate presentations of $A$ and $B$ so that the right adjoint for the compact lift of $f$ can be constructed from the right adjoints that are assumed to exist in (1).

We say that $\mathcal{V}$ has \textit{left uniform deductive interpolation} if, for any finite set of equations $\Delta(\bar{y}, \bar{z})$, there exists a finite set of equations $\Pi(\bar{y})$ such that for any set of equations $\Sigma(\bar{x}, \bar{y})$, $\Sigma \models \mathcal{V} \Delta$ iff $\Sigma \models \mathcal{V} \Pi$. Theorem 1 holds if one replaces
‘right’ by ‘left’ throughout. However, the property of left uniform interpolation is not entirely analogous to that of right uniform interpolation.

**Examples.** As above, Heyting algebras have left uniform deductive interpolation by [3]. It follows from the ‘left’ version of Theorem 1 that a locally finite variety $\mathcal{V}$ has left uniform deductive interpolation if, and only if, $\mathcal{V}$ has deductive interpolation and the compact lift of $F_{\mathcal{V}}(\pi) \hookrightarrow F_{\mathcal{V}}(\pi, \gamma)$ preserves intersections. In particular, we use these observations to give an algebraic proof that the variety of Brouwerian meet-semilattices does not have left uniform deductive interpolation.

One may now naturally wonder if an analogous result to Theorem 2 holds for left adjoints. It turns out that an additional condition is needed. We call a join-semilattice *dually Brouwerian* if the operation of binary join has a left residual.

**Theorem 3.** For any variety $\mathcal{V}$, the following are equivalent:

1. for any finite $\pi, \gamma$, the compact lift of $F_{\mathcal{V}}(\pi) \hookrightarrow F_{\mathcal{V}}(\pi, \gamma)$ has a left adjoint, and $K\text{Con}(F_{\mathcal{V}}(\pi))$ is a dually Brouwerian join-semilattice;
2. for any homomorphism $f: A \rightarrow B$ between finitely presented algebras of $\mathcal{V}$, the compact lift of $f$ has a left adjoint.

For the proof of this theorem, we first observe that, for any algebra $A$, the join-semilattice $K\text{Con}(A)$ is dually Brouwerian if, and only if, the compact lift of any surjective homomorphism $p: A \twoheadrightarrow B$ has a left adjoint. This characterization is subsequently combined with an argument similar to that in the proof of Theorem 2.

**References**