Bilattice Logic of Epistemic Action and Knowledge*

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The Logic of Epistemic Actions and Knowledge (EAK) has been introduced by Baltag, Moss and Solecki [4] as a framework for reasoning about knowledge in a dynamic setting. It is thus a language expansion of (classical) modal logic having, besides the usual modal operators that represent knowledge and beliefs of agents, dynamic operators used to represent the epistemic change that can be brought about by epistemic actions such as, e.g., announcements.

Formally, epistemic changes are modeled via the so-called *product update* construction on the Kripke-style models that constitute the relational semantics of EAK. Through the product update, a Kripke model encoding the current epistemic setup of a group of agents is replaced by an updated model, which encodes the setup of the agents after an epistemic action has taken place.

In [2, 3] product updates are dually characterized as a construction (called *epistemic update*) that transforms the complex algebra associated with a given Kripke model into the complex algebra associated with the model updated by means of an action structure. In this way EAK is endowed with an algebraic semantics that is dual to the relational one via a Jónsson-Tarski-type duality. Moreover, the methods of [2, 3] can be used to define a logic of Epistemic Actions and Knowledge on a propositional basis that is weaker than classical logic. This provides us with a more flexible logical formalism, which can be applied to a variety of contexts where classical reasoning may not be suitable. This line of research has been further pursued in [5, 6], which extends the mechanism of updates to the bilattice modal logic of [1], obtaining a *bilattice public announcement logic.*

In the present contribution we report on ongoing research that aims at further extending the methods of [5, 6] to introduce a suitable notion of product update on relational and algebraic models of bilattice modal logic, thus providing a semantics and a complete axiomatization for a bilattice-based Logic of Epistemic Action and Knowledge (BEAK).

Bilattice modal logic is a logic defined by Kripke models $\langle W, R, v\rangle$ in which both valuations and the accessibility relation $R\colon W \times W \to \mathsf{FOUR}$ take values into the four-element Belnap bilattice FOUR. The language of bilattice modal logic $\langle \wedge, \vee, \to, \neg, \Diamond, \mathsf{t}, \top, \mathsf{f}, \bot\rangle$ is essentially the same as that of classical modal logic (augmented with constants representing elements of FOUR), but the propositional connectives as well as the modal operator $\Diamond$ are interpreted using the algebraic operations of FOUR. This logic can be extended to define bilattice-

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based epistemic logics, for example a four-valued analogue of modal logic S5 (we refer to [1] for further details and motivation on bilattice modal logic).

We obtain the language of (single-agent)[3] BEAK by expanding bilattice modal logic with a dynamic modal operator $\langle \alpha \rangle$, where α is an action structure defined as below. Thus, for every formula $\varphi \in Fm$, we have that $\langle \alpha \rangle \varphi$ is also a formula. In our four-valued setting, an epistemic action is a structure $\alpha = (K, k, R_\alpha, \mathsf{Pre}_\alpha)$ where K is a finite non-empty set, $k \in K$, $R_\alpha : K \times K \to \mathsf{FOUR}$ and $\mathsf{Pre}_\alpha : K \to Fm$ is a map taking each point in K to a formula of BEAK (the precondition of the action).

Drawing inspiration from [2, 3], we introduce an algebraic semantics for BEAK via *intermediate structures*. For every modal bilattice $\langle \mathbf{B}, \Diamond \rangle$ (modal bilattices are the algebraic semantics of the bilattice modal logic introduced in [1]) and every action structure $\alpha = (K, k, R_\alpha, \mathsf{Pre}_\alpha)$, the non-modal reduct of the intermediate structure $\prod_\alpha \mathbf{B}$ is given by the direct power $\mathbf{B}^K$ (which is obviously an algebra in the same variety). The modal operator, however, is not defined on $\prod_\alpha \mathbf{B}$ in the standard way, but using a method that generalizes that of [2]. A special quotient of $\prod_\alpha \mathbf{B}$ is then taken, as an instance of the general construction introduced in [5, 6] to model public announcements in a bilattice setting. This is called *pseudo-quotient*, because it is obtained by means of a relation that is compatible with all bilattice connectives except for the $\Diamond$ operator. We note that the pseudo-quotient definition from [2] would not work in the bilattice setting, for it determines a relation that is already not compatible with a non-modal connective (the bilattice negation), and has therefore to be adapted, as detailed in [5, 6].

Our product and pseudo-quotient constructions allow us to define a suitable notion of algebraic models of BEAK. We then use the duality developed in [1] to obtain a relational semantics for the logic. Given a four-valued Kripke model $M = (W, R, V)$ and an action structure α, the intermediate structure $M \times \alpha$ is given by the coproduct $\coprod_\alpha M := (\coprod_K W, R \times R_\alpha, \coprod_K V)$, where $\coprod_K W$ is the $|K|$-fold coproduct of W (which is set-isomorphic to $W \times K$), $R \times R_\alpha$ is a four-valued relation on $\coprod_K W$ and $(\coprod_K V)(p) := \coprod_K V(p)$ for every atomic formula p. Finally, the update of M with the action structure α is the submodel $M^\alpha := (W^\alpha, R_\alpha, V^\alpha)$ of $\coprod_\alpha M$ the domain of which is the subset

$$W^\alpha := \{(w, j) \in \coprod_K W : M, w \models \mathsf{Pre}_\alpha(j)\}.$$

The constructions sketched above allow us to devise suitable interaction axioms between the dynamic modality and the other connectives of bilattice modal logic, which give us a Hilbert-style axiomatization of BEAK. Completeness with respect to algebraic models is obtained, as in [2, 3], via reduction to the static fragment of the logic; completeness with respect to the relational models follows then by duality.

[3] The multi-agent version of BEAK results from indexing modal operators with agents and interpreting relations (both on models and on action structures) over a set of agents.

References

1. Jung, A. and U. Rivieccio, *Kripke semantics for modal bilattice logic*, Proceedings of the 28th Annual ACM/IEEE Symposium on Logic in Computer Science, IEEE Computer Society Press, 2013, pp. 438–447.
2. Kurz, A. and A. Palmigiano, *Epistemic Updates on Algebras*, Logical Methods in Computer Science **9** (2013), pp. 1–28.
3. Ma, M., Palmigiano, A. and M. Sadrzadeh, *Algebraic semantics and model completeness for Intuitionistic Public Announcement Logic*, Annals of Pure and Applied Logic, **165** (2014), pp. 963–995.
4. A. Baltag, L. Moss and A. Solecki. The logic of public announcements, common knowledge, and private suspicions. CWI technical report SEN-R9922, 1999.
5. U. Rivieccio. Algebraic semantics for bilattice public announcement logic. A. Indrzejczak, J. Kaczmarek and M. Zawidzki (eds.), *Proceedings of Trends in Logic XIII (Lodz, Poland, 2-5 July 2014)*, Lodz University Press, 2014, pp. 199–215.
6. U. Rivieccio. Bilattice public announcement logic. R. Goré, B. Kooi and A. Kurucz (eds.), *Advances in Modal Logic*, Vol. 10, College Publications, 2014, p. 459–477.