K-theory of Modules as Model-theoretic Structures

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Motivated by the importance of the Grothendieck ring of varities in Kontsevich's theory of motivic integration, Krajicěk and Scanlon introduced the model-theoretic Grothendieck ring, $K_0(M)$, of a first-order structure M in [3] and studied definable version of some combinatorial principles in subsets of M^n definable with parameters, e.g., "definable pigeonhole principle".

Let $\mathcal{R}$ be a unital ring and $M_{\mathcal{R}}$ denote a right $\mathcal{R}$-module. Thinking of M as a model-theretic structure in the language of right $\mathcal{R}$-modules, Prest conjectured [6, Ch. 8, Conjecture A] that the Grothendieck ring of a nonzero $\mathcal{R}$-module is non-trivial. The first aim of the talk is to outline the proof of this conjecture. The proof proceeds by showing that such a Grothendieck ring is a quotient of a monoid ring [5, Theorem 5.2.3] using various tools from simplicial homology and semilattice theory.

The partial quantifier elimination result for complete theories of modules states that each subset of M^n definable with parameters is a finite boolean combination of subsets defined by positive primitive (*pp*)-formulas. Such a boolean combination is far from being unique, but one can obtain a canonical form using some techinques from (semi-)lattice theory. Such a canonical form is more general than Flenner and Guingona's Swiss cheese construction [1] which can be used to prove 1-dimensional elimination of imaginaries in VC-minimal theories.

For each $n \geq 1$, one can assign a "shape" to each definable subset of $M^n_{\mathcal{R}}$ using Euler characteristics of certain simplicial complexes. Various techniques from simplicial homology show that the shape is a family of valuations on the boolean algebra of definable sets, in the sense of Klain and Rota [2], each invariant under definable bijections. Bundling these valuations together gives that $K_0(M_{\mathcal{R}})$ is a quotient of the monoid ring $\mathbb{Z}[\mathcal{X}]$ by the invariants ideal $\mathcal{J}$, where $\mathcal{X}$ is the multiplicative monoid of *pp*-isomorphism classes of *pp*-subgroups and the invariant ideal $\mathcal{J}$ codes indices of pairs of *pp*-subgroups. Further if $M \neq 0$, then the ring $\mathbb{Z}$ can be embedded into $K_0(M_{\mathcal{R}})$ answering Prest's conjecture in the affirmative.

The definition of the Grothendieck ring of a first-order structure M coincides with the definition of Quillen's K_0 group of the symmetric monoidal groupoid $\mathrm{Def}(M)^{iso}$ of definable subsets of finite powers of M, under disjoint union, and definable bijections between them. (See [7] for the consturction of the K-theory of a symmetric monoidal categories.) This opens a new door to the study of algebraic invariants, namely algebraic K-groups, associated with the structure

M. This assignment is covariantly functorial on the elementary embeddings. Moreover it is the pairing on the symmetric monoidal groupoid defined using the product of definable sets that turns K_0 into a commutative ring. If M is a non-trivial finite structure, then Barratt-Priddy-Quillen-Segal theorem implies that $K_n(M)$ is the n^{th} stable homotopy group of spheres.

In the latter half of the talk, I will discuss this construction of the algebraic K-theory of a structure, give the complete description of the group, $K_1(V_F)$, of a vector space and state a conjecture about the structure of $K_1(M_{\mathcal{R}})$ for a module $M_{\mathcal{R}}$.

References

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