Generalizing Belnap's cut-elimination

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Display calculi. Nuel Belnap introduced the first display calculus, which he calls *Display Logic* [3], as a sequent system augmenting and refining Gentzen's basic observations on structural rules. Belnap's refinement is based on the introduction of a special syntax for the constituents of each sequent. Indeed, his calculus treats sequents $X \vdash Y$ where X and Y are so-called *structures*, i.e. syntactic objects inductively defined from formulas using an array of special connectives. Belnap's basic idea is that, in the standard Gentzen formulation, the comma symbol ',' separating formulas in the precedent and in the succedent of sequents can be recognized as a metalinguistic connective, of which the structural rules define the behaviour.

Belnap took this idea further by admitting not only the comma, but also several other connectives to keep formulas together in a structure, and called them *structural connectives*. Just like the comma in standard Gentzen sequents is interpreted contextually (that is, as conjunction when occurring on the left-hand side and as disjunction when occurring on the right-hand side), each structural connective typically corresponds to a pair of logical connectives, and is interpreted as one or the other of them contextually. Structural connectives maintain relations with one another, the most fundamental of which take the form of adjunctions and residuations. These relations make it possible for the calculus to enjoy the powerful property which gives it its name, namely, the *display property*.

Belnap's cut elimination metatheorem. In [3], a meta-theorem is proven, which gives a set of sufficient conditions in order for a sequent calculus to enjoy cut-elimination and subformula property. This meta-theorem captures the essentials of the Gentzen-style cut-elimination procedure, and is the main technical motivation for the design of Display Logic. The sufficient conditions in Belnap's meta-theorem are relatively easy to check, since most of them are verified by inspection on the shape of the rules. When Belnap's metatheorem can be applied, it provides a much smoother and more modular route to cut-elimination than the Gentzen-style proofs. Moreover, cut-elimination Belnap-style has the important advantage of being preserved under the addition of structural rules and introduction rules for new logical connectives,[1] whereas a Gentzen-style cut-elimination proof for the modified system cannot be deduced from the old one, but must be proved from scratch.

[1] Provided the rules in question verify certain conditions which we do not discuss here.

In a slogan, we could say that Belnap-style cut-elimination is to ordinary cut-elimination what canonicity is to completeness: indeed, canonicity provides a *uniform strategy* to achieve completeness. In the same way, the conditions required by Belnap's meta-theorem ensure that *one and the same* given set of transformation steps is enough to achieve cut elimination for any system satisfying them.[2] Various refinements and extensions of the original notion of display calculi exist in the literature, e.g. the proper display calculi in [16, Section 4.2] for the former, and [4] for the latter.

Contribution. The proposed contribution aims at reporting on the recent advances of a line of research [6–12] aimed at 'displaying dynamic logics'. Preliminary results of this line of research have been disseminated in the previous installment of TACL. The new advancements make it possible to overcome the hurdles specific to the settings of Baltag-Moss-Solecki's Dynamic Epistemic Logic (DEL) [1], Propositional Dynamic Logic (PDL) [14], and monotone modal logic [5, 13].

Methodology. The solutions to the specific technical difficulties of each logical system mentioned above require generalising Belnap's meta-theorem along different dimensions. Specifically, key to displaying DEL and PDL is the introduction of a *multi-type environment* for display calculi. This environment makes it possible to treat the parameters (actions, agents) of the modal connectives as terms in their own right. The difficulties in the treatment of the preconditions to the applicability of certain rules are dealt with by a suitable expansion of the language. Moreover, the display property is guaranteed by the introduction of certain structural connectives, referred to as *virtual adjoints* in [7], since they do not have any semantic interpretation. The price to pay to this language expansion is that one must prove separately that the resulting calculus is a conservative extension of the original logic. This was achieved in the case of the typed calculus for DEL with a relatively concise and smooth proof. The analogous proof for PDL is still an open problem.
The specific difficulty posed by monotone modal logic is the fact that its axiomatisation excludes the existence of the adjoints of the modal connectives. Rather than via virtual adjoints, the solution to this problem has been given in terms of a generalisation of Belnap's meta-theorem for calculi which do not enjoy the display property. Specifically, instead of it, the calculi are required to satisfy (a slight relaxation of) Sambin-Battilotti-Faggian's *visibility property* [2].

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