Proof by Order

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Considerable success has been enjoyed recently in providing general algebraic completeness proofs for cut-free sequent and hypersequent calculi with respect to classes of residuated lattices, the algebras of substructural logics (see [2]). These approaches, however, do not encompass "ordered group-like" structures such as (abelian) lattice-ordered groups and MV-algebras. Proof calculi have been defined for these classes in [6, 7, 4] but the completeness proofs are purely syntactic. The aim of the work reported here is to provide completeness results for such calculi using orderability theorems for groups and related structures.

Let us consider group terms as sequences, denoted Γ, Δ, of variables x and their inverses $\overline{x}$. That is, associativity is implicit, the inverse of $\Gamma = (s_1, \ldots, s_n)$ is $\overline{\Gamma} = (\overline{s}_n, \ldots, \overline{s}_1)$ where $\overline{\overline{x}} = x$, and the empty sequence represents the unit e. Then it is easily shown that the equation $\Gamma \approx e$ is valid in all abelian groups if and only if (henceforth iff) Γ is derivable using the rules

$$\frac{}{\Delta, \overline{\Delta}}\ (\textsc{id}) \qquad \frac{\Delta, \Gamma}{\Gamma, \Delta}\ (\textsc{cycle}) \qquad \frac{\Gamma \quad \Delta}{\Gamma, \Delta}\ (\textsc{mix}) \qquad \frac{\Gamma_1, \Delta_1, \Gamma_2, \Delta_2}{\Gamma_1, \Gamma_2, \Delta_1, \Delta_2}\ (\textsc{ex})$$

and valid in all groups iff Γ is derivable using these rules excluding (EX).

Consider now *lattice-ordered groups* (or *ℓ-groups*), defined as algebraic structures $(L, \wedge, \vee, \cdot, ^{-1}, e)$ where $(L, \wedge, \vee)$ is a lattice, $(L, \cdot, ^{-1}, e)$ is a group, and $\cdot$ preserves the order in both arguments. Every ℓ-group term is equivalent in ℓ-groups either to e or to a meet of joins of group terms, and checking validity of an ℓ-group equation amounts to checking the validity of a set of inequations of the form $e \leq \mathcal{G}$ where $\mathcal{G}$ is a non-empty set of group terms, understood as a join and written $\Gamma_1 \mid \ldots \mid \Gamma_n$.

Let GA (a one-sided version of the calculus from [6]) consist of the rules for abelian groups extended with

$$\frac{\mathcal{G} \mid \Gamma, \Delta}{\mathcal{G} \mid \Gamma \mid \Delta}\ (\textsc{split}) \qquad \frac{\mathcal{G}}{\mathcal{G} \mid \Gamma}\ (\textsc{ew})$$

It is easily shown that if $\mathcal{G}$ is derivable in GA, then $e \leq \mathcal{G}$ is valid in all abelian ℓ-groups. For the other direction, suppose that $\Gamma_1 \mid \ldots \mid \Gamma_n$ is not derivable in GA. Then the subsemigroup N of the ω-generated free abelian group $\mathbf{F}_{\mathcal{A}}$ generated by $\Gamma_1, \ldots, \Gamma_n$ (understood as elements of $\mathbf{F}_{\mathcal{A}}$) cannot contain e. So N defines a (strict) partial order on $\mathbf{F}_{\mathcal{A}}$ by setting $\Gamma < \Delta$ iff $\Gamma\overline{\Delta} \in N$ where in particular $\Gamma_i < e$ for $i = 1, \ldots, n$. But $\mathbf{F}_{\mathcal{A}}$ is torsion-free, so by a result of Fuchs (see [1] or [5]), this partial order extends to a total order, giving a (totally ordered) abelian ℓ-group where $e \leq \Gamma_1 \mid \ldots \mid \Gamma_n$ is not valid.

This pattern of reasoning generalizes to varieties of semilinear involutive commutative residuated lattices satisfying $x^n = nx$ for all $n \in \mathbb{N}^+$, covering both abelian ℓ-groups and Sugihara monoids (see [3, 7]). We suppose that there is a (one-sided) sequent calculus for the subreducts of such a variety restricted to the language of groups. Then it can be shown via a suitable orderability lemma that extending the sequent calculus with the hypersequent rules (SPLIT) and (EW) provides a calculus for the whole variety.

For the class of *totally ordered groups* (or *o-groups*), we obtain a calculus by extending the rules for groups with (SPLIT), (EW), and

$$\frac{\mathcal{G} \mid \Delta, \Gamma}{\mathcal{G} \mid \Gamma, \Delta}\ (\textsc{cycle}) \qquad \frac{\mathcal{G} \mid \Delta \quad \mathcal{G} \mid \overline{\Delta}}{\mathcal{G}}\ (*) \ \text{ where } \Delta \text{ is not group valid.}$$

In this case we make use of Ohnishi's theorem (see [5]) that a group G admits an o-group structure iff for all $a_1, \ldots, a_n \in G \setminus \{e\}$, there exist $\lambda_1, \ldots, \lambda_n \in \{-1, 1\}$ such that e is not in the normal subsemigroup generated by $a_1^{\lambda_1}, \ldots, a_n^{\lambda_n}$.

Similarly, removing the more general (CYCLE) rule provides a calculus for ℓ-groups, using Conrad's theorem that a group G admits a right-ordered group structure iff for all $a_1, \ldots, a_n \in G \setminus \{e\}$, there exist $\lambda_1, \ldots, \lambda_n \in \{-1, 1\}$ such that e is not in the subsemigroup generated by $a_1^{\lambda_1}, \ldots, a_n^{\lambda_n}$. This calculus is used in [4] to provide a new syntactic proof that the variety of ℓ-groups is generated by the ℓ-group of automorphisms of $\mathbb{R}$. It is also shown in [4] (but only syntactically) that $(*)$ may be replaced in these cases by the analytic rule

$$\frac{\mathcal{G} \mid \Gamma_1, \Delta_2 \quad \mathcal{G} \mid \Gamma_2, \Delta_1}{\mathcal{G} \mid \Gamma_1, \Delta_1 \mid \Gamma_2, \Delta_2}\ (\textsc{com})$$

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