Structural completeness in propositional logics of dependence

Rosalie Iemhoff and Fan Yang

Utrecht University, The Netherlands

1 Introduction

In this paper we prove that three of the main propositional logics of dependence are structurally complete with respect to a class of substitutions under which the logics are closed. As these logics are not structural, the notions of admissibility and structural completeness have to be considered relative to classes of substitutions with respect to which they are, as we do in this paper.

Dependence logic is a new logical formalism that characterizes the notion of "dependence" in social and natural sciences. First-order dependence logic was introduced by Väänänen [9] as a development of *Henkin quantifier* [2] and *independence-friendly logic* [3]. Recently, propositional dependence logic was studied and axiomatized in [10][8]. With a different motivation, Ciardelli and Roelofsen [1] introduced and axiomatized *inquisitive logic*, which turned out to be essentially equivalent to *propositional intuitionistic dependence logic*, a natural variant of propositional dependence logic. Dependency relations are characterized in these propositional logics of dependence by a new type of atoms $=(\vec{p},q)$, called *dependence atoms*. Intuitively, the atom specifies that *the proposition q depends completely on the propositions $\vec{p}$*. The semantics of these logics is called *team semantics*, introduced by Hodges [4][5]. The basic idea of this new semantics is that properties of dependence cannot be manifested in *single* valuations, therefore unlike the case of classical propositional logic, formulas in propositional logics of dependence are evaluated on *sets* of valuations (called *teams*) instead.

Propositional (intuitionistic) dependence logic as well as inquisitive logic characterize all downwards closed nonempty collections of teams. Therefore the three logics have the same expressive power. As a result of the feature of team semantics, the sets of theorems of these logics are closed under *flat substitutions*, but not closed under *uniform substitution*. In this paper, we prove that all admissible rules with respect to flat substitutions in these logics are derivable, that is, the three logics are structurally complete with respect to flat substitutions.

There is a close connection between inquisitive logic and certain intermediate logics. The set of theorems of the former equals the negative variant of Kreisel-Putnam logic (KP), which is equal to the negative variant of Medvedev logic (ML). The logic KP is not structurally complete, whereas ML is known to be structurally complete but not hereditarily structurally complete. An interesting corollary we obtain in this paper is that the negative variants of both ML and KP are hereditarily structurally complete with respect to negative substitutions. Related research has been carried out in [6][7].

Our methods are of a syntactic nature, but we do think that the problems could also be approached from an algebraic point of view, an issue that we hope will be addressed in the future.

2 Propositional logics of dependence and intermediate theories

The following grammars define the well-formed formulas of propositional dependence logic (PD) and propositional intuitionistic dependence logic (PID), respectively:

$$\varphi ::= p \mid \neg p \mid =(\vec{p},q) \mid \varphi \wedge \varphi \mid \varphi \otimes \varphi; \qquad \varphi ::= p \mid \bot \mid =(\vec{p},q) \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi.$$

Both PD and PID are defined in terms of *team semantics*, where a *team* is a *set* of valuations, i.e., a set of functions $v : \text{Prop} \to \{0,1\}$. We inductively define the notion of a formula φ being *true* on a team X, denoted by $X \models \varphi$, as follows:

- $X \models p$ iff for all $v \in X$, $v(p) = 1$;
- $X \models \neg p$ iff for all $v \in X$, $v(p) = 0$;
- $X \models \bot$ iff $X = \emptyset$;
- $X \models {=}(\vec{p}, q)$ iff for all $v, v' \in X$: $v(\vec{p}) = v'(\vec{p}) \implies v(q) = v'(q)$;
- $X \models \varphi \wedge \psi$ iff $X \models \varphi$ and $X \models \psi$;
- $X \models \varphi \otimes \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ such that $Y \models \varphi$ and $Z \models \psi$;
- $X \models \varphi \vee \psi$ iff $X \models \varphi$ or $X \models \psi$;
- $X \models \varphi \to \psi$ iff for any $Y \subseteq X$: $Y \models \varphi \implies Y \models \psi$.

Dependence atoms are definable in PID, and since PID without dependence atoms has the same syntax and semantics as inquisitive logic (InqL) [1], PID has the same expressive power as InqL. One can expand the syntax of PD and PID to include general dependence atoms ${=}(\vec{\varphi}, \psi)$ and arbitrary negations $\neg\varphi$. Actually, under this expansion the logics have the same expressive power as the original ones, and the deductive systems of the logics can be easily extended accordingly as well. We therefore identify PD and PID with their expansions in this paper. In summary, all of these logics (called *propositional logics of dependence*) have the same expressive power[1]. All these logics are strongly complete with respect to their deductive systems given in [10][1].

A substitution σ : Prop $\to$ Form is an L-*substitution* if the propositional logic L is closed under σ. An arbitrary substitution of the logic $\mathsf{L} \in \{\mathsf{PD}, \mathsf{PID}, \mathsf{InqL}\}$ is not necessarily an L-substitution. For example, $\vdash_{\mathsf{PID}} \neg\neg p \to p$,[2] but $\nvdash_{\mathsf{PID}} \neg\neg(p \vee \neg p) \to p \vee \neg p$. However, the logics of dependence are closed under *flat substitutions*, i.e., substitutions σ such that $\sigma(p)$ is *flat*[3] for all $p \in$ Prop.

Theorem 2.1 *For any logic* $\mathsf{L} \in \{\mathsf{PD}, \mathsf{PID}, \mathsf{InqL}\}$, *flat substitutions are* L-*substitutions.*

We call an L-substitution of a logic L that has implication and negation in its language a *negative substitution* if $\vdash_{\mathsf{L}} \sigma(p) \leftrightarrow \neg\neg\sigma(p)$ for all $p \in$ Prop. Flat substitutions are negative substitutions for PID and InqL. If L is an intermediate logic (i.e., a set of formulas closed under modus ponens and uniform substitution such that $\mathsf{IPC} \subseteq \mathsf{L} \subseteq \mathsf{CPC}$), then we call $\mathsf{L}^\neg = \{\varphi \mid \varphi^\neg \in \mathsf{L}\}$ the *negative variant* of L, where $(\cdot)^\neg$ is a substitution defined as $p^\neg = \neg p$ for all $p \in$ Prop. It is shown in [1] that $\mathsf{InqL} = \mathsf{KP}^\neg = \mathsf{ML}^\neg$. In general, $\mathsf{L}^\neg$ is the smallest intermediate theory[4] containing L and $\neg\neg p \to p$ for each $p \in$ Prop. The logic $\mathsf{L}^\neg$ is not closed under uniform substitution, but is closed under negative substitutions.

3 Flat formulas and projective formulas

Every consistent formula of the propositional logics of dependence considered in this paper is provably equivalent to a formula of the form $\bigvee_{i \in I} \Theta_{X_i}$, where each Θ_{X_i} is a formula in the language that defines a nonempty team X_i (up to its subteams). The formulas Θ_X turn out to

[1] L_1 and L_2 are said to have the *same expressive power* if for every L_1-formula φ, $\varphi \equiv \psi$ for some L_2-formula ψ, and vice versa, where $\varphi \equiv \psi$ iff $X \models \varphi \iff X \models \psi$ for all teams X.

[2] In PID, $\neg\varphi$ is an abbreviation of $\varphi \to \bot$.

[3] A formula φ is said to be *flat* if $X \models \varphi \iff \forall v \in X,\ \{v\} \models \varphi$ for all teams X.

[4] A set L of formulas is called an *intermediate theory* if $\mathsf{IPC} \subseteq \mathsf{L} \subseteq \mathsf{CPC}$ and L is closed under modus ponens.

form an important class of formulas. For one thing, a consistent formula φ is flat iff $\varphi \dashv\vdash \Theta_X$ for some nonempty team X. For another, formulas of the form Θ_X are *exactly* projective formulas for these propositional logics of dependence. Since some of these logics do not have implication in the language, and none of them is closed under uniform substitution, we modify the usual definition of projective formula. An L-formula φ is said to be $\mathcal{S}$-*projective* in L, where $\mathcal{S}$ is a set of L-substitutions, if there exists $\sigma \in \mathcal{S}$ such that

(1) $\vdash_{\mathsf{L}} \sigma(\varphi)$ **(2)** $\varphi, \sigma(\psi) \vdash_{\mathsf{L}} \psi$ and $\varphi, \psi \vdash_{\mathsf{L}} \sigma(\psi)$ for all L-formulas ψ.

Theorem 3.1 *Let* $\mathsf{L} \in \{\mathsf{PD}, \mathsf{PID}, \mathsf{InqL}\}$*,* $\mathcal{F}$ *the class of all flat substitutions, and* φ *a consistent* L*-formula. The following are equivalent:*

(1) $\varphi \dashv\vdash \Theta_X$ *for some nonempty team* X*;* ***(2)*** φ *is flat;* ***(3)*** φ *is* $\mathcal{F}$*-projective in* L*.*

Since $\mathsf{InqL} = \mathsf{KP}^{\neg}$, an interesting corollary of the above theorem is that for any intermediate logic $\mathsf{L} \supseteq \mathsf{KP}$, a consistent $\mathsf{L}^{\neg}$-formula φ is $\mathcal{N}$-projective in $\mathsf{L}^{\neg}$ iff $\varphi \dashv\vdash \neg\psi$ for some ψ, where $\mathcal{N}$ denotes the class of all negative substitutions.

4 Structural completeness

The logics considered in this paper are not closed under uniform substitution. To study admissible rules in these logics, we generalize the notion as follows. Let $\mathcal{S}$ be a set of L-substitutions of a logic L. A rule φ/ψ of L is said to be $\mathcal{S}$-*admissible* if $\vdash_{\mathsf{L}} \sigma(\varphi) \Longrightarrow \vdash_{\mathsf{L}} \sigma(\psi)$, for all $\sigma \in \mathcal{S}$. The logic L is said to be $\mathcal{S}$-*structurally complete* if every $\mathcal{S}$-admissible rule of L is derivable in L.

Theorem 4.1 PD*,* PID *and* InqL *are* $\mathcal{F}$*-structurally complete.*

Corollary 4.2 $\mathsf{KP}^{\neg}$*,* $\mathsf{ML}^{\neg}$ *are* $\mathcal{N}$*-hereditarily structurally complete, that is,* L *is* $\mathcal{N}$*-structurally complete, for any intermediate theory* L *such that* $\mathsf{KP}^{\neg}, \mathsf{ML}^{\neg} \subseteq \mathsf{L}$ *and* $\mathcal{N}$ *is a set of* L*-substitutions.*

References

1. Ciardelli, I., and Roelofsen, F. Inquisitive logic. *Journal of Philosophical Logic 40*, 1 (2011), 55–94.
2. Henkin, L. Some remarks on infinitely long formulas. In *Infinitistic Methods* (Warsaw, 1961), Proceedings Symposium Foundations of Mathematics, Pergamon, pp. 167–183.
3. Hintikka, J., and Sandu, G. Informational independence as a semantical phenomenon. In *Logic, Methodology and Philosophy of Science*, R. H. J. E. Fenstad, I. T. Frolov, Ed. Amsterdam: Elsevier, 1989, pp. 571–589.
4. Hodges, W. Compositional semantics for a language of imperfect information. *Logic Journal of the IGPL 5* (1997), 539–563.
5. Hodges, W. Some strange quantifiers. In *Structures in Logic and Computer Science: A Selection of Essays in Honor of A. Ehrenfeucht*, J. Mycielski, G. Rozenberg, and A. Salomaa, Eds., vol. 1261 of *Lecture Notes in Computer Science*. London: Springer, 1997, pp. 51–65.
6. Miglioli, P., Moscato, U., Ornaghi, M., Quazza, S., and Usberti, G. Some results on intermediate constructive logics. *Notre Dame Journal of Formal Logic 30*, 4 (1989), 543–562.
7. Prucnal, T. On two problems of Harvey Friedman. *Studia Logica 38*, 3 (1979), 247–262.
8. Sano, K., and Virtema, J. *Axiomatizing Propositional Dependence Logics*, http://arxiv.org/abs/1410.5038, 2014.
9. Väänänen, J. *Dependence Logic: A New Approach to Independence Friendly Logic*. Cambridge: Cambridge University Press, 2007.
10. Yang, F., and Väänänen, J. *Propositional Logics of Dependence and Independence, Part I*, http://arxiv.org/abs/1412.7998, 2014.