Jónsson-style Canonicity for ALBA-Inequalities (Unified Correspondence I)

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Canonicity. Canonicity is a fundamental notion in modal logic and other logics for which semantics based on relational structures are available, since it provides the main proof-path towards completeness results. Thanks to duality theory, canonicity can be investigated both in an algebraic and in a Kripke-semantic setting. In each of these settings, suitable syntactic identifications on given classes of formulas or inequalities are sought for, which guarantee the following preservation condition to hold for each element φ in the class:

$$\mathbb{S} \models \varphi \quad \Rightarrow \quad \mathbb{S}^\delta \models \varphi.$$

On the algebraic side of this account, $\mathbb{S}$ is an algebra and $\mathbb{S}^\delta$ is its *canonical extension* (cf. [17]), and on its relational structure side, $\mathbb{S}$ is a descriptive general frame, and $\mathbb{S}^\delta$ is its underlying frame.

Two approaches to canonicity. Most of the existing canonicity results have been obtained by means of one of the following two approaches, here respectively referred to as the *canonicity-via-correspondence* and *Jónsson-style*.

The *canonicity-via-correspondence* approach appears in [20] for the first time, and has also been pursued e.g. in [14,6]. The strategy of this approach follows an argument best illustrated by the following U-shaped diagram:

$$\begin{array}{ccc}
\mathbb{G} \Vdash \varphi & & \mathbb{F} \Vdash \varphi \\
\Updownarrow & & \\
\mathbb{F} \Vdash_{\mathbb{G}} \varphi & & \Updownarrow \\
\Updownarrow & & \\
\mathbb{F} \models_{\mathbb{G}} \mathrm{FO}(\varphi) & \Leftrightarrow & \mathbb{F} \models \mathrm{FO}(\varphi).
\end{array}$$

In the diagram above, $\mathbb{G} = (\mathbb{F}, \tau)$ is a descriptive general frame, $\mathbb{F}$ is the underlying Kripke frame of $\mathbb{G}$, and τ is its additional topological structure. This argument relies on the existence of a first-order sentence $\mathrm{FO}(\varphi)$, the *first-order correspondent* of φ, which holds of the Kripke frame $\mathbb{F}$ regarded as a first-order model iff φ is valid on $\mathbb{F}$, as shown in the right-hand side of the diagram. On its left-hand side, by definition, φ is valid on $\mathbb{G}$ iff φ is satisfied on $\mathbb{F}$ w.r.t. every admissible valuation (as the notation $\Vdash_{\mathbb{G}}$ represents). The proof succeeds if an analogous correspondence-type result holds restricted to admissible valuations, as represented by the vertical equivalence in the lower left-hand side of the diagram. Indeed, the bottom equivalence always holds, since the fact that $\mathrm{FO}(\varphi)$ holds does not depend on admissible valuations: in other words, $\mathrm{FO}(\varphi)$ cannot distinguish between $\mathbb{F}$ and $\mathbb{G}$. The canonicity results obtained following this approach are in essence byproducts of *correspondence theory*. The main contributions to this line of research have been the design of algorithms, such as SQEMA [4], for computing the first-order correspondents of large classes of

formulas. Another important contribution to this line of research is the syntactic characterization of the class of the so called *inductive formulas* [14], which properly extends the class of Sahlqvist formulas. Inductive formulas are shown to have first-order correspondent and be canonical.
The second approach to canonicity, referred to as *Jónsson-style*, originates in [16], and, independently, in [13], and has been pursued further in e.g. [12]. Its main features are its being purely algebraic, pursuing canonicity independently from correspondence, and its relying on the theory of *canonical extensions*. Up to this point, the literature displays a division of labour between the two approaches. Namely, algebras are used for studying canonicity independently from correspondence, and frames are used for canonicity-via-correspondence. In [6], this division of labour breaks down, and the results of [5] are generalized to perfect distributive lattices with operators by means of the algorithm ALBA. In [7], a purely algebraic algorithmic correspondence result in a non-distributive setting is given, via a version of ALBA which is sound on general lattices. The results in [6] and [7] show that correspondence, and hence canonicity-via-correspondence, can also be developed on algebras, and, with [8], they show that algebraic correspondence is grounded on the same order-theoretic principles guaranteeing the algebraic canonicity Jónsson-style, which forms the basis of *unified correspondence theory* [3]. This approach now covers a wide array of logics, e.g. regular modal logics [19], modal mu-calculus [1,2], hybrid logic [10], and has been applied to different issues, including the understanding of the relationship between different methodologies for obtaining canonicity results [19], or of the phenomenon of pseudocorrespondence [9], the dual characterizations of classes of finite lattices [11], and the identification of the syntactic shape of axioms which can be translated into structural rules of a properly displayable calculus [15].
Open issues. However, even if Jónsson-style and canonicity-via-correspondence use the same order-theoretic principles and the same setting of perfect algebras, they still look radically different. So it is natural to try and clarify how they relate to one another.
Contributions. The present talk reports on the results in [18], which address this open issue. In particular, they clarify the relationship between the two approaches to canonicity, and, as an application of the new insights, extend the Jónsson-style canonicity proof to the inductive and ALBA inequalities.
Technically, these results are made possible by generalizing the theory of canonical extensions of maps. In the literature, this theory studies extensions of maps $\mathbb{A} \to \mathbb{B}$ to maps $\mathbb{A}^\delta \to \mathbb{B}^\delta$. In [18], the theory of canonical extensions of maps $\mathbb{A} \to \mathbb{B}^\delta$ to maps $\mathbb{A}^\delta \to \mathbb{B}^\delta$ is developed, and is applied to the terms of the expanded language on which the algorithm ALBA runs, and on a further expansion of this language. This makes it possible to apply the Jónsson strategy for proving canonicity to inequalities in this expanded language, obtained by running ALBA on input inductive inequalities in the original DML language. The generalized theory of canonical extensions of maps guarantees that the terms in the expanded language have the right order-theoretic properties for the Jónsson strategy to be successful.

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