Sahlqvist theory for impossible worlds (Unified Correspondence V)

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Impossible worlds. The formalization of situations in which logical impossibilities are thinkable and sometimes even believable has been a key topic in modal logic since its onset, and has attracted the interest of various communities of logicians over the years. This specific imperfection of cognitive agency can be directly translated in the language of modal logic by stipulating that, for a given agent a, the formula $\Diamond_a \bot$ is not a contradiction, and hence, that the necessitation rule is not admissible. Impossible worlds have been introduced by Kripke in [15] in the context of his relational semantic account of modal logics, as an elegant way to invalidate the necessitation rule while retaining all other axioms and rules of normal modal logic, and hence to provide complete semantics for important non-normal modal logics such as such as Lemmon's systems E2-E4. More recently, impossible worlds have been used in close connection with counterfactual reasoning and paraconsistency. The reader is referred to [16] for a comprehensive survey on impossible worlds.

Impossible worlds and regular modal logics. The logics E2-E4 mentioned above are prominent examples of *regular modal logics*, which are classical modal logics (cf. [1]) in which the necessitation rule is not valid (equivalently, modal logics that do not contain $\Box\top$ as an axiom) but such that $\Box$ distributes over conjunction. Their lacking necessitation makes regular modal logics better suited than normal modal logics at the formalization of epistemic and deontic settings, since, as Lemmon argues, *nothing should be a scientific law or a moral obligation as a matter of logic.*
Notwithstanding the fact that the two variants of Kripke relational models (namely with and without impossible worlds) appeared almost at the same time, the state of development of their mathematical theory is not the same. In particular, although unsystematic correspondence results exist (viz. the ones in [15]), no Sahlqvist-type results are available. The proposed talk reports on the results in [18], extending state-of-the-art Sahlqvist theory to Kripke frames with impossible worlds.

Unified correspondence. In recent years, Sahlqvist theory has significantly broadened its scope, extending the benefits it originally imparted to modal logic to a wide range of logics which includes, among others, intuitionistic and distributive lattice-based (modal) logics [5], substructural logics [6], hybrid logics [9], and mu-calculus [2,3].

The breadth of this work has stimulated many and varied applications. Some are closely related to the core concerns of the theory itself, such as the understanding of the relationship between different methodologies for obtaining canonicity results [17], or of the phenomenon of pseudocorrespondence [8]. Other, possibly surprising applications include the dual characterizations of classes of finite lattices [10], and the identification of the syntactic shape of axioms which can be translated into structural rules of a properly displayable calculus [12]. These and other results (cf. [7]) form the body of a theory called *unified correspondence* [4], a framework within which correspondence results can be formulated and proved abstracting away from specific logical signatures, and only in terms of the order-theoretic properties of the algebraic interpretations of logical connectives.

Focus of the proposed talk. In [18], we apply the unified correspondence approach to obtain Sahlqvist-type canonicity and correspondence results for regular modal logics on different propositional bases. We mainly focus on two aspects: *Jónsson-style canonicity* and *algorithmic correspondence and canonicity.*

Jónsson-style canonicity builds on the theory of canonical extensions, originating in [14]. This method has been pioneered in [13], where the canonicity of Sahlqvist formulas of classical normal modal logic was proven in a purely algebraic way. Interestingly, this method does not rely on Sahlqvist correspondence. Jónsson's method for canonicity was also adopted in [11], in the setting of distributive modal logic (DML).

We analyze Jónsson strategy as laid out in [11]. The conclusion of our analysis is that, rather than the properties of being normal operators or dual operators, the actual engine of the Sahlqvist mechanism is given by the additivity and multiplicativity of the algebraic interpretations of the logical connectives. In this sense, the regular setting provides a kind of conceptual completion for Jónsson-style canonicity.

As to the algorithmic correspondence for regular modal logics, we introduce an adaptation, referred to as ALBA^r, of the calculus ALBA to regular modal logic (on weaker than classical bases). We define the class of inductive inequalities in the regular setting. Similarly to the inductive inequalities defined in other settings, inductive DLR-inequalities properly and significantly extend Sahlqvist inequalities, while sharing their most important properties, namely the fact that the (regular) modal logics generated by them are strongly complete w.r.t. the class of Kripke frames defined by their first-order correspondent. We show that ALBA^r succeeds on every inductive DLR-inequality.

Finally, the previous results are applied to obtain the strong completeness of Lemmon's logics E2-E5 w.r.t. elementary classes of Kripke frames with impossible worlds, and the defining first-order conditions are effectively computed via ALBA^r.

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