Subordinations, closed relations, and compact Hausdorff spaces

Guram Bezhanishvili[1], Nick Bezhanishvili[2], Sumit Sourabh[2], Yde Venema[2]

[1] Department of Mathematical Sciences, New Mexico State University, Las Cruces NM 88003

[2] Institute for Logic, Language and Computation, University of Amsterdam, P.O. BOX 94242, 1090 GE Amsterdam, The Netherlands

By the celebrated Stone duality [11], the category of Boolean algebras and Boolean homomorphisms is dually equivalent to the category of Stone spaces (compact Hausdorff zero-dimensional spaces) and continuous maps. De Vries [12] generalized Stone duality to the category of compact Hausdorff spaces and continuous maps. Objects of the dual category are complete Boolean algebras B with a binary relation $\prec$ (called by de Vries a compingent relation) satisfying certain conditions that resemble the definition of a proximity on a set [10].

Another generalization of Stone duality is central to modal logic. We recall that modal algebras are Boolean algebras B with a unary function $\Box : B \to B$ preserving finite meets, and modal spaces (descriptive frames) are Stone spaces X with a binary relation R satisfying certain conditions. Stone duality then generalizes to a duality between the categories of modal algebras and modal spaces (see, e.g., [5, 9, 3]).

The dual of a modal algebra $(B, \Box)$ is the modal space (X, R), where X is the Stone dual of B (the space of ultrafilters of B), while the binary relation $R \subseteq X \times X$ is the Jónsson-Tarski dual of $\Box$ [8]. Unlike the modal case, in de Vries duality we do not split the dual space of $(B, \prec)$ in two components, the Stone dual of B and the relation R. Instead we work with the space of "$\prec$-closed" filters which are maximal with this property.

The aim of this paper is to develop an alternative "modal-like" duality for de Vries algebras, in which we do split the dual space of a de Vries algebra $(B, \prec)$ in two parts: the Stone dual of B and the dual of $\prec$. If X is the de Vries dual of $(B, \prec)$, then the Stone dual Y of B is the Gleason cover of X [1]. We show that the irreducible map $\pi : Y \to X$ gives rise to what we call an irreducible equivalence relation R on Y, which is the dual of $\prec$. It follows that compact Hausdorff spaces are in 1-1 correspondence with pairs (Y, R), where Y is an extremally disconnected compact Hausdorff space and R is an irreducible equivalence relation on Y. We call such pairs Gleason spaces, and introduce the category of Gleason spaces, where morphisms are relations rather than functions, and composition is not relation composition. We prove that the category of Gleason spaces is equivalent to the category of compact Hausdorff spaces and continuous maps, and is dually equivalent to the category of de Vries algebras and de Vries morphisms, thus providing an alternate "modal-like" duality for de Vries algebras.

For this we introduce a general concept of a subordination $\prec$ on a Boolean algebra B. A *subordination* on a Boolean algebra B is a binary relation $\prec$ satisfying:

(S1) $0 \prec 0$ and $1 \prec 1$;
(S2) $a \prec b, c$ implies $a \prec b \wedge c$;
(S3) $a, b \prec c$ implies $a \vee b \prec c$;
(S4) $a \leq b \prec c \leq d$ implies $a \prec d$.

A subordination can be seen as a generalization of the modal operator $\Box$. If we generalize the dual modal operator $\Diamond$ the same way, then we arrive at the well-known concept of a precontact relation and a precontact algebra [6, 7]. Subordinations also correspond to quasi-modal operators of [4]. Other interesting examples of subordinations (satisfying additional conditions) are de Vries' compingent relations and lattice subordinations of [2].

Since subordinations, precontact relations, and quasi-modal operators are definable from each other, all three concepts are represented dually by means of closed relations on Stone spaces. In addition, we provide duality for the corresponding morphisms, thus establishing a full categorical duality for the categories of interest.

References

1. Bezhanishvili, G.: Stone duality and Gleason covers through de Vries duality. Topology Appl. 157(6), 1064–1080 (2010)
2. Bezhanishvili, G.: Lattice subordinations and Priestley duality. Algebra Universalis 70(4), 359–377 (2013)
3. Blackburn, P., de Rijke, M., Venema, Y.: Modal logic. Cambridge University Press, Cambridge (2001)
4. Celani, S.A.: Quasi-modal algebras. Mathematica Bohemica 126(4), 721–736 (2001)
5. Chagrov, A., Zakharyaschev, M.: Modal logic. The Clarendon Press, New York (1997)
6. Dimov, G., Vakarelov, D.: Topological representation of precontact algebras. In: MacCaull, W., Winter, M., Düntsch, I. (eds.) Relational Methods in Computer Science, Lecture Notes in Computer Science, vol. 3929, pp. 1–16. Springer Berlin Heidelberg (2006)
7. Düntsch, I., Vakarelov, D.: Region-based theory of discrete spaces: a proximity approach. Ann. Math. Artif. Intell. 49(1-4), 5–14 (2007)
8. Jónsson, B., Tarski, A.: Boolean algebras with operators. I. Amer. J. Math. 73, 891–939 (1951)
9. Kracht, M.: Tools and techniques in modal logic. North-Holland Publishing Co., Amsterdam (1999)
10. Naimpally, S.A., Warrack, B.D.: Proximity spaces. Cambridge University Press, London (1970)
11. Stone, M.H.: The theory of representations for Boolean algebras. Trans. Amer. Math. Soc. 40(1), 37–111 (1936)
12. de Vries, H.: Compact spaces and compactifications. An algebraic approach. Ph.D. thesis, University of Amsterdam (1962)