Monadic Fragments of Modal Predicate Logics

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In [10] Wajsberg proved that the modal logic $\mathsf{S5}$ axiomatizes the one-variable fragment (which we will call the *monadic* fragment) of classical predicate logic (QCPC). Prior [8] introduced an intuitionistic analog of $\mathsf{S5}$, known as MIPC, and Bull [4] proved that MIPC axiomatizes the monadic fragment of intuitionistic predicate logic (QIPC). A more transparent proof was later given by Ono and Suzuki [7] using a modification of the famous Henkin construction. The lattice of extensions of MIPC was studied in [1–3] and the correspondence between extensions of MIPC and extensions of QIPC in [7, 9]. Our goal is to generalize these results to the modal setting.

Let $\mathcal{QL}_M$ be the modal predicate language, and let QK be the least set of formulas of $\mathcal{QL}_M$ containing all theorems of QCPC, the axiom $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, and closed under uniform substitution, modus ponens, generalization ($\frac{\varphi}{\forall x\varphi}$), and $\Box$-necessitation ($\frac{\varphi}{\Box\varphi}$). We define a *modal predicate logic* to be an extension M of QK closed under these rules. Let BM denote the extension of a modal predicate logic M by the Barcan formula $\forall x\Box\varphi \to \Box\forall x\varphi$.

Let $\mathcal{L}_M$ be the propositional modal language and let $\mathcal{L}_{MM}$ be the enrichment of $\mathcal{L}_M$ by the monadic operator $\forall$. Let mK be the least set of formulas of $\mathcal{L}_{MM}$ containing all axioms of the normal modal logic K for $\Box$, the $\mathsf{S5}$ axioms for $\forall$, the bridge axiom $\Box\forall\varphi \to \forall\Box\varphi$, and closed under substitution, modus ponens, $\Box$-necessitation, and $\forall$-necessitation ($\frac{\varphi}{\forall\varphi}$). We define a *monadic modal logic* (*mm-logic*) to be an extension L of mK closed under these rules. Let bL denote the extension of a mm-logic L by the *Barcan formula* $\forall\Box\varphi \to \forall\Box\varphi$.

We define a translation T from $\textbf{Form}(\mathcal{L}_{MM})$ to $\textbf{Form}(\mathcal{QL}_M)$ inductively by first associating to each propositional letter p a unary predicate $P(x)$ and then setting

$$\underset{\text{for prop. letters } p}{T(p) = P(x)}\ , \quad \underset{\text{for } \bullet=\vee,\wedge}{T(\varphi \bullet \psi) = T(\varphi) \bullet T(\psi)}, \quad \underset{\text{for } *=\neg,\Box}{T(*\varphi) = *T(\varphi)}, \quad T(\forall\varphi) = \forall x T(\varphi).$$

For a mm-logic $\mathsf{L} \supseteq \mathsf{mK}$, we define $\Phi(\mathsf{L}) = \mathsf{QK} + \{T(\varphi) : \mathsf{L} \vdash \varphi\}$ to be the modal predicate logic which extends QK by the translations of all theorems of L. Similarly, for a modal predicate logic $\mathsf{M} \supseteq \mathsf{QK}$, we define $\Psi(\mathsf{M}) = \mathsf{mK} + \{\varphi : \mathsf{M} \vdash T(\varphi)\}$ to be the mm-logic which extends mK by all formulas of $\mathcal{L}_{MM}$ whose translations are theorems of M.

Lemma 1. *For* $\mathsf{L} \supseteq \mathsf{mK}$ *and* $\mathsf{M} \supseteq \mathsf{QK}$ *we have*

1. $\Phi(\mathsf{L}) \subseteq \mathsf{M}$ *iff* $\mathsf{L} \subseteq \Psi(\mathsf{M})$.

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2. $\Psi(\Phi(\mathsf{L})) \supseteq \mathsf{L}$ *with equality iff* $\mathsf{L} = \Psi(\mathsf{M})$ *for some* M.
3. $\mathsf{M} \supseteq \Phi(\Psi(\mathsf{M}))$ *with equality iff* $\mathsf{M} = \Phi(\mathsf{L})$ *for some* L.

Definition 1. *We call* $\mathsf{L} \supseteq \mathsf{mK}$ the monadic fragment *of a modal predicate logic* $\mathsf{M} \supseteq \mathsf{QK}$ *if* $\mathsf{L} \vdash \varphi$ *iff* $\mathsf{M} \vdash T(\varphi)$, *and denote this relationship by* $\langle \mathsf{L}; \mathsf{M} \rangle$.

To determine whether $\langle \mathsf{L}; \mathsf{M} \rangle$ for a pair of logics $\mathsf{L} \supseteq \mathsf{mK}$ and $\mathsf{M} \supseteq \mathsf{QK}$ we will need to develop a correspondence between semantics for mm-logics and modal predicate logics, which will allow us to prove results concerning extensions of mK similar to those of Ono and Suzuki [7].

An mK-*frame* is a Kripke frame $\mathfrak{F} = \langle W, R, E \rangle$ where E is an equivalence relation and $RE(w) \subseteq ER(w)$ for all $w \in W$. Recall that a predicate Kripke frame with expanding (constant) domains is a frame $\mathfrak{F} = \langle W, R, D \rangle$, where D assigns to each $w \in W$ a set D_w of objects, and wRv implies $D_w \subseteq D_v$ ($D_w = D_v$) for all $w, v \in W$. Since there's no readily apparent way to translate between mK-frames and predicate Kripke frames, we employ a smaller class of frames arising from product frames.

We recall that a *product* [6, p. 222] of Kripke frames $\mathfrak{F}_1 = \langle W_1, R_1 \rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2 \rangle$ is a frame $\mathfrak{F}_1 \times \mathfrak{F}_2 = \langle W_1 \times W_2, R_h, R_v \rangle$ where $(u_1, v_1) R_h (u_2, v_2)$ iff $u_1 R_1 u_2$ and $v_1 = v_2$, and $(u_1, v_1) R_v (u_2, v_2)$ iff $u_1 = u_2$ and $v_1 R_2 v_2$. An mK-frame $\langle W, R, E \rangle$ is an *expanding relativized product frame (erp-frame)* [6, p. 432] if $\mathfrak{F}$ is a subframe of a product frame $\mathfrak{F}_1 \times \mathfrak{F}_2$ and for all $(w_1, w_2) \in W$ and $u \in W_1$, if $w_1 R u$ then $(u, w_2) \in W$. We will call $\mathfrak{F}_1$ and $\mathfrak{F}_2$ the *underlying frames* of $\mathfrak{F}$.

We can now easily translate back and forth between erp-frames and predicate Kripke frames, which provides a basis for our translation theorem. For an erp-frame $\mathfrak{F} = \langle W, R, E \rangle$ we associate a predicate Kripke frame $\mathfrak{F}^\dagger = \left\langle W^\dagger, R^\dagger, D \right\rangle$, where $\left\langle W^\dagger, R^\dagger \right\rangle = \langle W_1, R_1 \rangle$, and $D_w = \{v \in W_2 : (w, v) \in W\}$ for each $w \in W^\dagger$. We can then define a truth relation by $(\mathfrak{F}^\dagger, w) \vDash p^v_x$ iff $(\mathfrak{F}, (w, v)) \vDash p$, where φ^v_x is used to denote the formula obtained from φ by replacing every free occurrence of x by v . Similarly, if we start with a predicate Kripke frame $\mathfrak{F} = \langle W, R, D \rangle$, we associate an erp-frame $\mathfrak{F}^\times = \langle W^\times, R^\times, E^\times \rangle$ with underlying frames $\langle W, R \rangle$ and $\langle V, V \times V \rangle$, where $V = \bigcup_{w \in W} D_w$ and $W^\times = \{(u, v) \in W \times V : v \in D_u\}$. We define a truth relation by $(\mathfrak{F}^\times, (w, v)) \vDash p$ iff $(\mathfrak{F}, w) \vDash p^v_x$ and arrive at the following theorem.

Theorem 1. *1. If* $\mathfrak{F}$ *is an erp-frame and* $\varphi \in \mathbf{Form}(\mathcal{L}_{MM})$, *then* $(\mathfrak{F}, (w, v)) \vDash \varphi$ *iff* $(\mathfrak{F}^\dagger, w) \vDash (T(\varphi))^v_x$.
2. If $\mathfrak{F}$ *is a predicate Kripke frame and* $\varphi \in \mathbf{Form}(Q\mathcal{L}_M)$, *then* $(\mathfrak{F}, w) \vDash (T(\varphi))^v_x$ *iff* $(\mathfrak{F}^\times, (w, v)) \vDash \varphi$.

To establish our results we need completeness of some basic monadic modal systems with respect to either product or erp-frames. Completeness for these systems is given by the following.

Theorem 2. *1.* [6, Thm. 9.10] mK *is complete with respect to the class of all erp-frames, and for* $\mathsf{L} \in \{\mathsf{K4}, \mathsf{S4}, \mathsf{S5}\}$, mL *is complete with respect to the class of all erp-frames for which* R *is transitive (*K4*)/ a quasi-order (*S4*)/ an equivalence relation (*S5*).*

2. [6, Cor. 5.10] bK *is complete with respect to the class of all product frames, and for* $\mathsf{L} \in \{\mathsf{K4}, \mathsf{S4}\}$[1]*,* bL *is complete with respect to the class of all product frames for which* R *is transitive (*$\mathsf{K4}$*)/ a quasi-order (*$\mathsf{S4}$*).*

It is possible to give a simpler proof of Theorem 2 using a modified Henkin construction, which is more in keeping with Ono and Suzuki's translation theorem ([7, Theorem 3.5]).We then have the following theorem.

Theorem 3. *Let* $\mathsf{L} \supseteq \mathsf{mK}$ *be a mm-logic complete with respect to a class* $\{\mathfrak{F}_i\}_{i \in I}$ *of erp-frames. If* $\mathsf{M} \supseteq \mathsf{QK}$ *is complete with respect to* $\{\mathfrak{F}_i^\dagger\}_{i \in I}$*, then* $\langle \mathsf{L}; \mathsf{M} \rangle$*.*

As a consequence of Theorems 2 and 3, we obtain:

Corollary 1. *For* $\mathsf{L} \in \{\mathsf{K}, \mathsf{K4}, \mathsf{S4}, \mathsf{S5}\}$ *we have* $\langle \mathsf{mL}; \mathsf{QL} \rangle$ *and for* $\mathsf{L} \in \{\mathsf{K}, \mathsf{K4}, \mathsf{S4}\}$[2] *we have* $\langle \mathsf{bL}; \mathsf{BL} \rangle$*.*

The bimodal logic $\mathsf{mS4}$ was first considered by Fischer Servi [5]. She extended the Gödel translation of IPC to $\mathsf{S4}$ to a translation of formulas φ of MIPC to formulas φ^t of $\mathsf{mS4}$, and proved that $\mathsf{MIPC} \vdash \varphi$ iff $\mathsf{mS4} \vdash \varphi^t$. The proof required that $\mathsf{QS4} \vdash T(\varphi)$ when $\mathsf{mS4} \vdash \varphi$, but whether or not the other implication holds was left as an open problem. Corollary 1 gives the other implication, and also allows for a simplified version of her proof that $\mathsf{MIPC} \vdash \varphi$ iff $\mathsf{mS4} \vdash \varphi^t$.

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[1] Note that $\mathsf{bS5} = \mathsf{mS5}$.
[2] Note that $\mathsf{BS5} = \mathsf{QS5}$.