Interpolation in Brouwer logics determined by k-branching nets of clusters

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Abstract. We study normal extensions of the modal Brouwer logic **KTB** which have nice semantical characterization. They are logics determined by nets of clusters with bounded number of branching. It occurred that the studied logics are Kripke complete and have finite model property. Then we investigate the logics with respect to interpolation. It is known that the logics **KTB** has interpolation. We prove that there are very few locally finite logics within the family of $NEXT(\mathbf{KTB})$ with interpolation.

We continue research on Brouwerian modal logics having Craig interpolation property. In paper [2] we describe two-step transitive Brouwerian modal logics without interpolation. Now, we study Brouwerian logics not being n-transitive at all with respect to the interpolation property.

We examine the Brouwer modal logic **KTB** and its normal extensions, which are determined by a class of Kripke frames equipped with a tolerance relation and having special forms. Each reflexive and symmetric Kripke frame may be divided into blocks of tolerance. Blocks of tolerance are linearly ordered if one of them has non-empty intersection with at most two other blocks. If one block of tolerance sees at most k other blocks then we call such a Kripke frame as k-branching net of clusters.

Referring to Brouwerian linear logics, they may be axiomatized by adding the following axiom (see [3]):

$$(3') \ := \Box p \vee \Box(\Box p \to \Box q) \vee \Box((\Box p \wedge \Box q) \to r).$$

It was proved in [3] and [6] that

Theorem 1 *All logics from $NEXT(\mathbf{KTB.3'})$ are Kripke complete and have f.m.p. The cardinality of $NEXT(\mathbf{KTB.3'})$ is uncountably infinite.*

The lattice of $NEXT(\mathbf{KTB.3'})$ is extremely complicated. However, its structure may be characterized using the technics of splitting , see [4]. Referring to Kripke frames in which blocks of tolerance are connected with some bounded number of others, we may generalize the axiom (3') as follows:

$$(n') \ := \Box p_1 \vee \Box(\Box p_1 \to \Box p_2) \vee ... \vee \Box((\Box p_1 \wedge ... \wedge \Box p_n) \to \Box p_{n+1})$$

and study the logics $NEXT(\mathbf{KTB.n'})$.

It will be proved in [5], that:

Theorem 2 *For the given $n \in \mathbb{N}$ the logics from $NEXT(\mathbf{KTB.n'})$ are Kripke complete and have f.m.p.*

Let us add that our approach is more general than enriching the Brouwer logic with the axioms
$alt_n := \Box p_1 \vee \Box(p_1 \to p_2) \vee ... \vee \Box((p_1 \wedge ... \wedge p_n) \to p_{n+1})$, $n > 2$.

In our talk, we describe locally finite logics from $NEXT(\mathbf{KTB.n'})$ having Craig interpolation property or at least interpolation property for deducibility. To do this, we characterize Halldén complete logics. Let us remind the basic definitions.

Definition 1 *A logic L has the Craig interpolation property (CIP) if for every implication $\alpha \to \beta$ in L, there exists a formula γ (interpolant for $\alpha \to \beta$ in L) such that $\alpha \to \gamma \in L$ and $\gamma \to \beta \in L$ and $Var(\gamma) \subseteq Var(\alpha) \cap Var(\beta)$.*

Definition 2 *A logic L has the interpolation property for deducibility (IPD) if for any α, β such that $\alpha \vdash_L \beta$, there exists a formula γ such that $\alpha \vdash_L \gamma \in L$ and $\gamma \vdash_L \beta \in L$ and $Var(\gamma) \subseteq Var(\alpha) \cap Var(\beta)$.*

The condition (IPD) is weaker than (CIP).

Definition 3 *A logic L is Halldén complete if*

$$\varphi \vee \psi \in L \text{ implies } \varphi \in L \text{ or } \psi \in L$$

for all φ and ψ containing no common variables.

There is an important connection between the Craig interpolation property and Halldén completeness of modal logics. It is presented in the following lemma due to G. F. Schumm [9]:

Lemma 1 *If L has only one Post-complete extension and is Halldén-incomplete, then interpolation fails in L.*

On the other hand we know that [1]:

Theorem 3 *If a modal logic L is determined by one Kripke frame, which is homogeneous, then L is Halldén complete.*

The above theorem may be strengthened to an equivalence for some special frames [7]:

Theorem 4 *Let $\mathfrak{F} := \langle W, R \rangle$ be KTB-Kripke frame, which is finite and connected. Logic $L(\mathfrak{F})$ is Halldén complete iff the frame $\mathfrak{F}$ is homogenous.*

Among Halldén complete and locally finite logics from $NEXT(\mathbf{KTB.n'})$ we characterize the logics having (CIP) and(IPD).

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