A Representation Theorem for System P

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We prove a representation theorem for non-monotonic inference relations that are defined over elements of some Boolean algebras and obey the rules of System P. An inference relation is represented by a closure operator on the Stone space of the Boolean algebra. This representation theorem generalizes and gives new insights into existing completeness theorems for System P.

Let A be a Boolean algebra. A *non-monotonic inference relation on* A is a binary relation $\mathrel{|\!\sim}$ on the carrier of A which satisfies the axioms of System P:

- $a \mathrel{|\!\sim} a$ (Id)
- If $a \mathrel{|\!\sim} c$ and $c \leq d$ then $a \mathrel{|\!\sim} d$ (RW)
- If $a \mathrel{|\!\sim} c$ and $a \mathrel{|\!\sim} d$ then $a \mathrel{|\!\sim} c \wedge d$ (And)
- If $a \mathrel{|\!\sim} b$ and $a \mathrel{|\!\sim} c$ then $a \wedge b \mathrel{|\!\sim} c$ (CM)
- If $a \mathrel{|\!\sim} c$ and $b \mathrel{|\!\sim} c$ then $a \vee b \mathrel{|\!\sim} c$ (Or)

The standard semantics for System P is given by a poset $\leq$ on some set W. Assuming that we have a Boolean algebra homomorphism $[\![\cdot]\!] : A \to \mathcal{P}W$ we define $a \mathrel{|\!\sim} c$ to hold if for all $w \in [\![a]\!]$ there is a $v \in [\![a]\!]$ with $v \leq w$ such that $u \in [\![c]\!]$ for all $u \leq v$ with $u \in [\![a]\!]$. If $\leq$ is well-founded this is equivalent to setting $a \mathrel{|\!\sim} c$ if the $\leq$-minimal elements of $[\![a]\!]$ are all in $[\![c]\!]$. This clause captures the intuition that a non-monotonic inference from a to c holds if the most relevant instances of a are instances of c.

System P is applied as a framework for defeasible reasoning in artificial intelligence [4]. It is also the non-nested fragment of a conditional logic developed in philosophy and linguistics [5, 7]. In this context the restriction to non-nested formulas is not essential for most properties of the logic. Moreover, the same semantics is used in belief revision [2, 1].

Completeness proofs for System P, seen as inferences system over Boolean formulas, with respect to its order semantics are provided by [4] and [7]. These proofs are technical and it is not clear how they can be seen as extensions of the Stone duality between Boolean algebras and Stone spaces. The reason for these difficulties is a mismatch between the order semantics and System P. It it noticed in [7] that the addition of a condition, called coherence, to System P greatly simplifies the completeness proof. This condition is also considered in the context of AGM belief revision where it is the postulate (K-8r) [6]. The problem with the coherence condition is that it is rather complex and thus not expressible in the language of conditional logic. By increasing the expressivity of the language and assuming an analogue of coherence [8] obtains a general representation result. The language of [8] contains an unary operator f whose semantic interpretation maps $[\![a]\!] \subseteq W$ to the the set $[\![f(a)]\!] \subseteq W$ of $\leq$-minimal elements of $[\![a]\!]$. This is strictly more expressive than conditional logic and unnatural in applications to defeasible reasoning.

In our approach we represent a non-monotonic inference relation on a Boolean algebra with a closure operator on the Stone space of the algebra. This generalizes the order semantics as one can take the closure operator to map a set to its upset in a poset. The upsets in a poset are closed under unions which corresponds to the fact that this closure operator preserves arbitrary joins. Our semantics is a weakening of the order semantics in that we do not require the preservation of arbitrary joins. In this way we remove the mismatch between the System P and its order semantics by weakening the semantics instead of adding the coherence condition on the algebraic side.

Our main result is that for every non-monotonic inference relation $\mid\!\sim$ on a Boolean algebra A there is a closure operator cl on the Stone space of A such that $a \mid\!\sim c$ iff $\bar{a} \subseteq \mathsf{cl}(\bar{a} \cap \bar{c})$, where we write $\bar{u}$ for the clopen corresponding to an element u of A. In the statement of this result we identify the elements a and c of A with the corresponding clopen set in the stone space.

An crucial observation behind this result is to define for every clopen set A, corresponding to an element a of the algebra, the closed set $M_A = \bigcap\{\bar{c} \mid a \mid\!\sim c\}$. One can think of M_A as the set of minimal elements in A. The closure operator cl is then defined to be the closure operator corresponding to the meet-semilattice of all sets X which satisfy that $A \subseteq X$ for every clopen A with $M_A \subseteq X$. A crucial property of this construction is that for all clopens A and subsets X of the Stone space we have that

$$M_A \subseteq X \quad \text{iff} \quad A \subseteq \mathsf{cl}(A \cap X).$$

In the case where cl takes upsets in some poset this condition characterizes M_A as the minimal elements of A.

We also obtain the completeness of System P with respect to posets. For this aim we prove that every Stone space with a closure operator resulting from the representation is the image of a closure operator that preserves arbitrary joins under a continuous function that preserves the validity of conditionals. One can then see that this closure operator results from a poset.

In the finite case we obtain a proper duality by using a suitable notion of morphism and observing that the closure operators corresponding to an inference relation are precisely the antimatroids over the atoms of the Boolean algebra. Antimatroids are a generalization of posets that provide a combinatorial abstraction of the notion of a convex set [3]. We are currently trying to extend the duality to the infinite case.

References

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