Quasi-primal Cornish algebras

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Abstract. Varieties generated by quasi-primal algebras are a natural generalisation of Boolean algebras and play an important role in the interface between logic and universal algebra. A Cornish algebra is a bounded distributive lattice equipped with a family of unary operations each of which is either an endomorphism, and so models a strong form of modal operator, or a dual endomorphism, and so models a De Morgan negation. We characterise quasi-primal Cornish algebras. The results yield as a special case a recent result by Davey, Nguyen and Pitkethly describing quasi-primal Ockham algebras. Our characterisation is in terms of the Priestley dual of the algebra.

Keywords: quasi-primal algebra, Cornish algebra, Ockham algebra, Priestley duality

In the late 1940s, Tarski proved that the first-order theory of Boolean algebras is decidable [7]. This can be restated as the *the first-order theory of the variety generated by the two-element Boolean algebra is decidable*. In the early 1970s, it was realised (independently) by Pixley [5, 6] and Werner [8] that many of the important properties of the variety of Boolean algebras stem from the fact that the *ternary discriminator* function t, given by

$$t(x, y, z) = \begin{cases} x, & \text{if } x \neq y, \\ z, & \text{if } x = y, \end{cases}$$

is a term function of the two-element Boolean algebra. (It is very easy to check that $t(x, y, z) := ((x \wedge z) \vee y') \wedge (x \vee z)$ yields the ternary discriminator on the two-element Boolean algebra.)

A finite algebra is called *quasi-primal* if it has the ternary discriminator as a term function. In the late 1970s, Werner extended Tarski's result by showing that every variety generated by a quasi-primal algebra has a decidable first-order theory. Ten years later, this result played an important role in McKenzie and Valeriote's characterisation of locally finite varieties with a decidable theory [4]. (See Werner [9] for a detailed discussion of quasi-primal algebras.) Given their

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importance, it is natural to seek descriptions of the quasi-primal algebras in familiar classes of algebras. Here we characterise quasi-primal Cornish algebras.

An algebra $\mathbf{A} = \langle A; \vee, \wedge, F^{\mathbf{A}}, 0, 1\rangle$ is a *Cornish algebra* if $\mathbf{A}^\flat := \langle A; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice, $F = F_- \mathbin{\dot\cup} F_+$, and each $f^{\mathbf{A}} \in F^{\mathbf{A}}$ is an endomorphism of $\mathbf{A}^\flat$ if $f \in F_+$ and is a dual endomorphism of $\mathbf{A}^\flat$ if $f \in F_-$. When $f \in F_-$, the operation $f^{\mathbf{A}}$ models a negation that satisfies De Morgan's laws. The operations $f^{\mathbf{A}}$, for $f \in F_+$, model strong modal operators which preserve $\wedge$ as well as $\vee$; the *next* operator of linear temporal logic is an example. Cornish algebras are a natural generalisation of *Ockham algebras*, which have been extensively studied [1]. Ockham algebras are the special case of Cornish algebras in which $F = F_- = \{f\}$.

Recently, Davey, Nguyen and Pitkethly [3] characterised quasi-primal Ockham algebras. Here we give an external characterisation of quasi-primal Cornish algebras and derive from it completely internal sufficient conditions for a Cornish algebra to be quasi-primal. Our results yield the Davey–Nguyen–Pitkethly result for Ockham algebras as a special case. Both the external necessary-and-sufficient conditions and the internal sufficient conditions are expressed in terms of the Priestley dual of the algebra.

Let $H\colon \boldsymbol{\mathcal{D}} \to \boldsymbol{\mathcal{P}}$ and $K\colon \boldsymbol{\mathcal{P}} \to \boldsymbol{\mathcal{D}}$ be the Priestley duality functors between the categories $\boldsymbol{\mathcal{D}}$ of bounded distributive lattices and $\boldsymbol{\mathcal{P}}$ of Priestley spaces. Using Priestley duality, each finite Cornish algebra $\mathbf{A}$ is, up to isomorphism, of the form $\mathbf{A} = K(\mathbb{W})$ where $\mathbb{W}$ is a finite *Cornish space*, that is, a structure $\mathbb{W} := \langle W; F^{\mathbb{W}}, \leqslant\rangle$ such that $\langle W; \leqslant\rangle$ is a finite ordered set, and $F^{\mathbb{W}} = \{\, f^{\mathbb{W}} \mid f \in F\,\}$ is a set of unary operations on W with $f^{\mathbb{W}}$ order-preserving for each $f \in F_+$, and order-reversing for each $f \in F_-$. *Ockham spaces* arise in the special case when $F = F_- = \{f\}$.

Since a quasi-primal algebra must be simple, it follows easily that if $\mathbf{A} = K(\mathbb{W})$ is quasi-primal, then the Cornish space $\mathbb{W}$ has no non-empty proper substructures. It is less obvious that a quasi-primal Cornish algebra must have $F_- \neq \varnothing$.

Lemma 1. *Let $\mathbf{A} = K(\mathbb{W})$ be a quasi-primal Cornish algebra. Then $F^- \neq \varnothing$ and $\mathbb{W}$ has no non-empty proper substructures.*

Our external characterisation of quasi-primal Cornish algebras is in terms of pairs of jointly surjective morphisms.

Theorem 2. *Let $\mathbf{A} = K(\mathbb{W})$ be a finite Cornish algebra. Then $\mathbf{A}$ is quasi-primal if and only if*

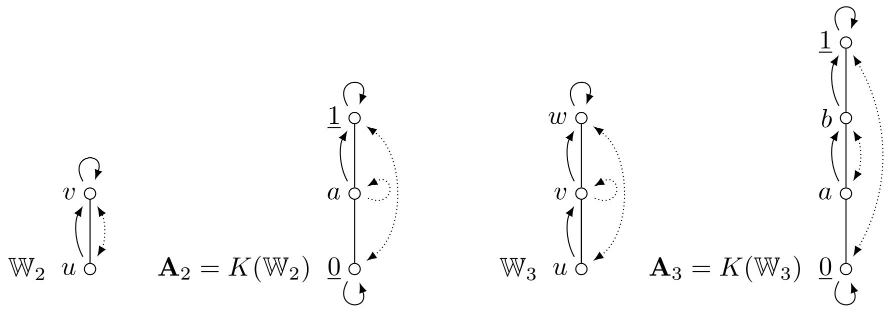
(i) *$\mathbb{W}$ has no proper non-empty substructures, and*
(ii) *for every finite Cornish space $\mathbb{P}$ and every pair of jointly surjective morphisms $\varphi_0, \varphi_1\colon \mathbb{W} \to \mathbb{P}$, if there exists $a, b \in W$ with $\varphi_0(a) \leqslant \varphi_1(b)$ or $\varphi_0(a) \geqslant \varphi_1(b)$, then there exists $c, d \in W$ with $\varphi_0(c) = \varphi_1(d)$.*

We can now give our internal sufficient conditions on $\mathbb{W}$ for $\mathbf{A} = K(\mathbb{W})$ to be quasi-primal. Since each operation symbol $f \in F$ belongs to either F_+ or F_-, we can extend the $+/-$ labelling to the set T of unary terms in the natural way.

Theorem 3. *Let* $\mathbf{A} = K(\mathbb{W})$ *be a finite Cornish algebra. Then* $\mathbf{A}$ *is quasi-primal provided* $\mathbb{W}$ *has the following properties:*

(i) $\mathbb{W}$ *has no proper non-empty substructures, and*
(ii) *there exists a unary term* $t \in T^{-}$ *such that, for all* $w \in W$*, the* $t^{\mathbb{W}}$*-orbit of* w *eventually reaches an odd cycle.*

As an application of Thm. 3, below we see the start of an infinite family of quasi-primal Cornish algebras: $f \in F_{+}$ are indicated by solid lines and $f \in F_{-}$ are indicated by dotted lines.



The characterisation of quasi-primal Ockham algebras by Davey, Nguyen and Pitkethly [3] can easily be derived from our results.

Theorem 4. *Let* $\mathbb{W} := \langle W; f^{\mathbb{W}}, \leqslant\rangle$ *be a finite Ockham space. Then the Ockham algebra* $\mathbf{A} = K(\mathbb{W})$ *is quasi-primal if and only if* $|W|$ *is odd,* $\langle W; \leqslant\rangle$ *is an antichain and* $f^{\mathbb{W}}$ *is a cyclic permutation of* W*.*

References

1. Blyth, T.S., Varlet, J.C.: Ockham algebras. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York (1994)
2. Cornish, W.H.: Antimorphic Action: Categories of Algebraic Structures with Involutions or Anti-endomorphisms. Research and Exposition in Mathematics, Heldermann, Berlin (1986)
3. Davey, B.A., Nguyen, L.T., Pitkethly, J.G.: Counting relations on Ockham algebras. Algebra Universalis (in press)
4. McKenzie, R., Valeriote, M.: The Structure of Decidable Locally Finite Varieties. Progress in Mathematics, 79. Birkhäuser, Boston (1989)
5. Pixley, A.F.: Functionally complete algebras generating distributive and permutable classes. Math. Z. 114, 361–372 (1970)
6. Pixley, A.F.: The ternary discriminator function in universal algebra. Math. Ann. 191, 167–180 (1971)
7. Tarski, A.: Arithmetical classes and types of Boolean algebras. Bull. Amer. Math. Soc. 55, 64 (1949)
8. Werner, H.: Eine Charakterisierung funktional vollständiger Algebren. Arch. Math. (Basel) 21, 381–385 (1970)
9. Werner, H.: Discriminator Algebras. Studien zur Algebra und Ihre Anwendungen, Band 6, Akademie-Verlag, Berlin (1978)