A General Extension Theorem for Complete Partial Orders

Peter Schuster[1] and Daniel Wessel[2]

[1] Department of Computer Science, University of Verona
Strada le Grazie 15, 37134 Verona, Italy
[2] Department of Mathematics, University of Trento
Via Sommarive 14, 38123 Povo (TN), Italy
daniel.wessel@unitn.it

1 Introduction

An invocation of Zorn's Lemma (ZL) often takes place within an indirect proof of a universal statement. Supposing towards a contradiction that there be any counterexample, the maximal counterexample provided by ZL helps—by a "one-step" argument [1]—to the desired contradiction. Crucially though, this "one-step" does not depend on maximality, and in fact a more general method is hovering in the background, which a priori is not limited to hypothetical counterexamples only.

As a consequence, this and related proof patterns can sometimes [2,3,4,5,8,9] be turned into direct proofs with Open Induction (OI) [7] as an equivalent of ZL. We now bring this approach to a somewhat unexpected type of application: to extension theorems such as the ones going back to Helly, Hahn and Banach as well as to Baer's Criterion for whether a module is injective.

To this end, we distill a General Extension Theorem (GET) for complete partial orders, the intended meaning being that the poset under consideration consists of partial extensions of which one is to be proved total. The principal hypothesis of GET encodes, inspired by a trick due to Northcott [6], the "one-step" argument from before: that there be a function extending each partial extension by any potential element of its domain. As compared with the typical indirect proof with ZL of an extension theorem, GET postulates the existence of a total extension rather than a maximal one.

2 Extension Patterns

Common ground in the usual setting of an extension theorem not only is the aforementioned "one-step" principle, but also the domain of any partial extension has to be kept track of in a reasonable way. Abstractly, this is pinned down as follows:

Definition 1. *Let E be a partially ordered set. An* extension pattern *for E is given by a set S, a monotone mapping $D : E \to \mathcal{P}(S)$, and a function f :*

$E \times S \to E$, satisfying the extension property

$$\forall e \in E \; \forall x \in S \; \big(\; e \leq f(e,x) \; \wedge \; x \in D(f(e,x)) \; \big).$$

The subset $D(e) \subseteq S$ is called domain *of $e \in E$. An element e of E with $D(e) = S$ is called* total*.*

The intended meaning of S is that of a set of extension data by which elements of E may be extended. Moreover, in analogy to the example of a partial order of functions, the domain informs about the “extent” of such an element.

Applying ZL, it is readily seen that a complete partial order (cpo) has a total element for every extension pattern. Indeed, every maximal element equals any of its extensions in the pattern, and thus is total.

Here is our *General Extension Theorem.*

GET *Every cpo with extension pattern has a total element.*

Proving GET by ZL naturally resembles the respective proofs of its instances. Likewise, proving GET by OI can be done with the predicate of “being totally extendable”. While this clearly works for proving specific extension theorems, we now have stated it abstractly in terms of an extension pattern.

Given a partially ordered set E and an arbitrary set S, by constantly assigning $D(e) = S$ to $e \in E$, we see that total elements need not be maximal; totality is defined with respect to a given pattern! On the other hand, there is an extension pattern for which the notions of totality and maximality do coincide: here a cpo E works as extension set for itself by

$$D(e) = \{ \, x \in E : e \not< x \, \} \quad \text{and} \quad f(e,x) = \begin{cases} x & \text{if } e < x, \\ e & \text{otherwise} \end{cases}$$

for all $e, x \in E$. A total element for this pattern clearly is maximal. ZL (formulated appropriately) thus follows from GET, whence they are equivalent in an appropriate fragment of ZF set theory.

While GET is an immediate and direct consequence of both ZL and OI, the converse implication requires the Law of Excluded Middle (LEM). In fact, LEM considerably adds to the applicability of GET, since to define an extension function often requires to decide on the domain of that element which is to be extended.

Attempting to fit some prominent examples into extension patterns, e.g. to deduce the Axiom of Choice (AC) directly from GET, has brought us to make an interesting move: the family of non-empty sets for which a choice function is sought needs to be replaced by a family of pointed sets. The following adaption of the extension pattern allows for extending by more involved extension data and defines totality only relative to a given surjection.

Definition 2. *Let E be a partially ordered set. A* relative extension pattern *for E is given by a surjective function $\pi : S \to T$, a monotone mapping $D : E \to$*

$\mathcal{P}(S)$, and a function $f : E \times S \to E$, satisfying the relative extension property

$$\forall e \in E \; \forall x \in S \; \big(\; e \leq f(e,x) \;\; \wedge \;\; \pi(x) \in \pi[D(f(e,x))] \; \big).$$

Here an element e of E is called total if $\pi[D(e)] = T$.

The resulting Relative Extension Theorem (rGET) is directly equivalent to GET.

rGET *Every cpo with relative extension pattern has a total element.*

References

1. J. L. Bell. Zorn's lemma and complete boolean algebras in intuitionistic type theories. *The Journal of Symbolic Logic*, 62(4):1265–1279, Dec. 1997.
2. Ulrich Berger. A computational interpretation of open induction. In F. Titsworth, editor, *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science*, pages 326–334. IEEE Computer Society, 2004.
3. Francesco Ciraulo, Davide Rinaldi, and Peter Schuster. Logical completeness via open induction. Preprint, University of Leeds, 2014.
4. Thierry Coquand. Constructive topology and combinatorics. In *Constructivity in Computer Science (Summer Symposium San Antonio, TX, June 19–22, 1991 Proceeding)*, volume 613 of *Lecture Notes in Computer Science*, pages 159–164. Springer Berlin Heidelberg, 1992.
5. Matthew Hendtlass and Peter Schuster. A direct proof of wiener's theorem. In S. Barry Cooper, Anuj Dawar, and Benedikt Löwe, editors, *How the World Computes. Turing Centenary Conference and 8th Conference on Computability in Europe*, volume 7318 of *Lecture Notes in Computer Science*, pages 293–302. Springer Berlin Heidelberg, 2012.
6. D. G. Northcott. *An Introduction to Homological Algebra.* Cambridge University Press, 1960.
7. Jean-Claude Raoult. Proving open properties by induction. *Information Processing Letters*, 29(1):19–23, 1988.
8. Davide Rinaldi and Peter Schuster. A Universal Krull-Lindenbaum Theorem. Technical report, University of Leeds, 2013.
9. Peter M. Schuster. Induction in algebra: a first case study. *Logical Methods in Computer Science*, 9(3:20), 2013.