Towards a Riesz Representation Theorem for Finite Heyting Algebras

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To establish a defensible notion of probability for events evaluated in a non-classical logic L, whose algebraic semantics is given by some variety $\mathbb{L}$, it is useful to setup a conceptual framework consisting of the following connections, which generalise the connections between classical propositional logic, Boolean algebras, and classical probability theory.

1. A Stone-like duality theorem between the category of $\mathbb{L}$-algebras and their homomorphisms and a suitable category of spaces.
2. A de Finetti-like coherence criterion defining probability of events subject to the logical constraints imposed by L.
3. A Riesz-like representation theorem to express probabilities as integral measures over the dual spaces of $\mathbb{L}$-algebras.

Focusing on propositional intuitionistic logic, it is well-known that its algebraic semantics is given by the variety $\mathbb{H}$ of Heyting algebras, and that the category $\mathbb{H}$ is dually equivalent to the category of Esakia spaces [4], that is Priestley spaces [6] such that the downward closure of each clopen subset is also clopen. So, item (1) of the conceptual framework exposed above is fully established. By contrast, no common accepted notion seems to exist for (2) and (3).

In this work we propose a Riesz-like integral representation for the case of dual spaces of finite Heyting algebras, that is, for finite posets and order-preserving open maps between them. Our representation is based upon a kind of reduction to the case of prelinear Heyting algebras, as we are going to illustrate.

Prelinear Heyting algebras are Heyting algebras A satisfying the additional identity $(x \to y) \vee (y \to x) = \top$, for all $x, y \in A$. Prelinear Heyting algebras are known as Gödel algebras by many-valued logicians. The associated logic, that is, the propositional logic G having prelinear Heyting algebras as its corresponding algebraic semantics, is called Gödel or Gödel-Dummett propositional logic.

The variety $\mathbb{G}$ of Gödel algebras is dually equivalent, as a category, to the profinite completion of the category FF of *finite forests* and order-preserving open maps between them. Recall that a forest is a poset such that the downward closure of each one of its elements is totally ordered by restriction. Then FF is dually equivalent to finitely presented Gödel algebras.

As $\mathbb{G}$ is locally finite, the notions of finitely generated, finitely presented, and finite algebras coincide. The duality between the full subcategory $\mathbb{G}_{fin}$ of finite

Gödel algebras and FF has been used in [2], together with the characterisation of the n-generated free Gödel algebra as an algebra of suitable functions $f\colon [0,1]^n \to [0,1]$ explained in [5], to give a Riesz-like representation theorem for finite Gödel algebras.

The notion of probability measure that arises for a finite Gödel algebra A corresponds to a functional $s\colon A \to [0,1]$ satisfying certain natural and expected properties of integrals, such as additivity and monotonicity, and of probability measures, such as normalisation. But it must be observed that it is necessary to postulate an additional property that is far from being obvious.

This property translates in the following rule, where P is a probability:

$$\frac{\vdash \varphi \to \psi \qquad P(\psi) \leq P(\varphi)}{P(\psi \to \varphi) = 1}\,.$$

Notice that this rule is automatically valid for classical logic and classical probability, while in Gödel logic there are functionals $s\colon A \to [0,1]$ satisfying all other requested natural properties that fail it.

In [1] and [3] the authors define a de Finetti-like betting scheme, over events observed during time, that matches the definition of probability given by the Riesz-like representation theorem for finite Gödel algebras, thus fully settling the notions (1), (2), and (3) for this case.

We show how to generalise the Riesz-like theorem for finite Gödel algebras to the case of finite Heyting algebras, in such a way to define a class of functionals $s\colon A \to [0,1]$ satisfying all properties requested for the Gödel algebras case. This is obtained by working on the dual side, and considering the smallest forest that surjects onto a given finite poset via a map satisfying certain properties (that we call *semi-openness*). Dually, this corresponds to embedding the lattice reduct of a Heyting algebra into a Gödel algebra, in an universal way, according to a suitable definition aimed at preserving as much as possible of the Heyting algebraic structure. This construction defines the desired integral representation as the restriction to the original Heyting algebra of the functional providing the integral representation in the enveloping Gödel algebra.

References

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