De Vries powers: A generalization of Boolean powers for compact Hausdorff spaces

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For an algebra A and a Boolean algebra B, the *Boolean power of A by B* is the algebra $C(X, A_{\text{disc}})$ of all continuous functions from the Stone space X of B to A, where A is given the discrete topology and the operations of A are lifted to $C(X, A_{\text{disc}})$ pointwise (see, e.g., [1, 6]). For convenience, we also refer to $C(X, A_{\text{disc}})$ as the *Boolean power of A by X*. Since their introduction by Foster [9, 10], Boolean powers have proved to be a very useful tool in universal algebra and model theory; see [5, 1, 6] for a detailed history and key results.

The Boolean power construction is not useful for an arbitrary compact Hausdorff space. If X is compact and A is discrete, each $f \in C(X, A_{\text{disc}})$ is finitely valued, and gives a partition of X into finitely many clopen sets. So if there are not enough clopens in X, then $C(X, A_{\text{disc}})$ is not representative enough. For example, if $X = [0, 1]$, then $C(X, A_{\text{disc}}) = A$. Our goal is to generalize the Boolean power construction in such a way that it encompasses compact Hausdorff spaces.

One of the most natural generalizations of Stone duality to compact Hausdorff spaces is de Vries duality [12]. We recall that a binary relation $\prec$ on a Boolean algebra B is a *proximity* if it satisfies:
(1) $1 \prec 1$.
(2) $a \prec b$ implies $a \leq b$.
(3) $a \leq b \prec c \leq d$ implies $a \prec d$.
(4) $a \prec b, c$ implies $a \prec b \wedge c$.
(5) $a \prec b$ implies $\neg b \prec \neg a$.
(6) $a \prec b$ implies there is $c \in B$ such that $a \prec c \prec b$.
(7) $a \neq 0$ implies there is $0 \neq b \in B$ such that $b \prec a$.

A *de Vries algebra* is a complete Boolean algebra with a proximity. By de Vries duality, each compact Hausdorff space X gives rise to the de Vries algebra $(\mathcal{RO}(X), \prec)$, where $\mathcal{RO}(X)$ is the complete Boolean algebra of regular open subsets of X. The proximity is given by $U \prec V$ iff $\mathsf{Cl}(U) \subseteq V$, where Cl is the closure in X. Moreover, each de Vries algebra $(B, \prec)$ is isomorphic to the de Vries algebra $(\mathcal{RO}(X), \prec)$ for a unique compact Hausdorff space X.

We use de Vries duality to define the de Vries power of an algebra by a compact Hausdorff space the same way Stone duality is used to define the Boolean power of an algebra by a Stone space. Let A be a totally ordered algebra, let X be a compact Hausdorff space, and let $f : X \to A$ be a finitely valued function. We call f *normal* if $f^{-1}(\uparrow a)$ is regular open in X for each $a \in A$, where $\uparrow a = \{b \in A : a \leq b\}$. If $A = \mathbb{R}$, this is equivalent to f being normal in the sense of [8]. Let $FN(X, A)$ be the set of finitely valued normal functions from

X to A. We introduce the concept of *normalization*, and show that normalization lifts the operations of A to $FN(X, A)$ and makes $FN(X, A)$ an ordered algebra. In addition, $FN(X, A)$ has a canonical proximity given by $f \prec g$ iff $f^{-1}(\uparrow a) \prec g^{-1}(\uparrow a)$ in $\mathcal{RO}(X)$ for each $a \in A$. We call the pair $(FN(X, A), \prec)$ the *de Vries power of* A *by* X. If X is a Stone space, this construction yields the Boolean power construction.

Our main goal is to axiomatize de Vries powers of a totally ordered integral domain, thus including such classic cases as $\mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$. Our results generalize several results in the literature. Boolean powers of $\mathbb{Z}$ were studied by Ribenboim [11]. They turn out to be exactly the Specker ℓ-groups introduced and studied by Conrad [7]. On the other hand, Boolean powers of $\mathbb{R}$ are the Specker $\mathbb{R}$-algebras introduced and studied in [4]. Boolean powers of a commutative ring were studied in [3], and were shown to coincide with the class of Specker A-algebras introduced in that article. Over an integral domain A, the Specker A-algebras are precisely the torsion-free idempotent generated A-algebras.

For a totally ordered domain A, we enrich the concept of a Specker A-algebra to that of a proximity Specker A-algebra, and show that a de Vries power of a totally ordered domain is precisely a proximity Specker A-algebra that is also a Baer ring. We prove that each proximity Specker A-algebra $(S, \prec)$ can be represented as a dense subalgebra of $(FN(X, A), \prec)$ for a unique (up to homeomorphism) compact Hausdorff space X, and that $(S, \prec)$ is isomorphic to $(FN(X, A), \prec)$ iff S is a Baer ring.

To describe the space X associated with a proximity Specker A-algebra $(S, \prec)$, we extend the notion of an end of a de Vries algebra to obtain the notion of an end ideal of $(S, \prec)$. With the appropriate topology on this set of end ideals we get a space homeomorphic to the space of ends of the de Vries algebra of idempotents of S with induced proximity relation.

By introducing the notion of a proximity morphism, we extend de Vries duality to show that the category of compact Hausdorff spaces is dually equivalent to that of proximity Baer Specker A-algebras. Full proofs of these results are available in [2].

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