Complete axiomatizations of lexicographic sums and products of modal logics

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We consider two natural operations on modal logics — lexicographic (or ordered) sums and products. Like "usual" product of modal logics, the lexicographic sum and the lexicographic product of modal logics are defined semantically — via corresponding operation on their frames.

Investigation of lexicographic products of modal logics was started by the first author in [2], where numerous completeness results were obtained. Lexicographic sums of modal logics appeared in many contexts: for example, sums of provability logics were axiomatized by L. Beklemishev [3]; the second author applied sums to investigate complexity of modal logics [4]; decidability and the finite model property of sums were investigated by S. Babenyshev and V. Rybakov [1].

We present new general completeness results for lexicographic sums and products of modal logics. In particular, it follows that in many cases these operations lead to the same logics (Theorems 1 and 2), and the resulting logic is the fusion extended by three certain Sahlqvist formulas. Theorem 3 describes more difficult case when we need infinitely many extra axioms to axiomatize lexicographic products; in particular, it gives the axiomatization of the lexicographic square of the minimal logic **K**.

Definitions. For the sake of simplicity, we consider operations on monomodal logics.

Definition 1 *Let* $\mathsf{I} = (I, S)$ *be a frame,* $\{\mathsf{F}_i = (W_i, R_i) \mid i \in I\}$ *be a family of frames. The* lexicographic *(or* ordered*)* sum $\sum_{\mathsf{I}} \mathsf{F}_i$ *is the frame* (W, R_+, S_+), *where* W *is the disjoin sum* $\sum_I W_i = \{(w, i) \mid i \in I, w \in W_i\}$, *and*

$$(w,i)R_+(u,j) \iff i = j \ \&\ wR_iu,$$
$$(w,i)S_+(u,j) \iff iSj.$$

If for all i $\mathsf{F}_i = \mathsf{F}$, *we write* $\mathsf{F} \rhd \mathsf{I}$ *for* $\sum_{\mathsf{I}} \mathsf{F}_i$; *the frame* $\mathsf{F} \rhd \mathsf{I}$ *is called* the lexicographic product of frames F and I.

⋆ The research is partially supported by the Russian Foundation for Basic Research and French National Centre for Scientific Research (RFBR–CNRS grant no. 14-01-93105).

Definition 2 *For logics L_1, L_2, put*

$$\sum_{L_2} L_1 = \textsc{Log}(\{\sum_{\mathsf{I}} \mathsf{F}_i \mid \mathsf{I} \models L_2,\ \{\mathsf{F}_i \mid i \textit{ in } \mathsf{I}\} \models L_1\}),$$
$$L_1 \rhd L_2 = \textsc{Log}(\{\mathsf{F} \rhd \mathsf{I} \mid \mathsf{F} \models L_1, \mathsf{I} \models L_2\}).$$

Completeness. A modal formula is *closed*, if it does not contain propositional variables. By a *closed sentence* we mean the standard translation of a closed modal formula. A logic L is *Horn axiomatizable*, if the class of all its frames is the class of all first-order models of a theory consisting of strict universal Horn sentences and closed sentences. The standard systems $\mathbf{K}, \mathbf{T}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}$ are examples of Horn axiomatizable logics.

Consider the following bimodal formulas:

$$\alpha = \Box_2 p \to \Box_1 \Box_2 p, \quad \beta = \Box_2 p \to \Box_2 \Box_1 p, \quad \gamma = \Diamond_2 p \to \Box_1 \Diamond_2 p.$$

$L_1 * L_2$ denotes the fusion of logics L_1 and L_2.

Theorem 1. *Consider unimodal logics L_1 and L_2. Suppose $L_1 * L_2 + \{\alpha, \beta, \gamma\}$ is Kripke complete, L_2 is Horn axiomatizable. Then $\sum\limits_{L_2} L_1 = L_1 * L_2 + \{\alpha, \beta, \gamma\}$.*

Corollary 1. *Let L_1 and L_2 be canonical unimodal logics, L_2 Horn axiomatizable. Then $\sum\limits_{L_2} L_1 = L_1 * L_2 + \{\alpha, \beta, \gamma\}$.*

The following theorem is a generalization of [2, Proposition 32].

Theorem 2. *If L_1 and L_2 are Horn axiomatizable Kripke complete unimodal logics and $\Diamond\top \in L_1$, then $L_1 \rhd L_2 = L_1 * L_2 + \{\alpha, \beta, \gamma\}$.*

Definition 3 *For unimodal logics L_1, L_2 put*

$$L_1 \rhd_s L_2 = L_1 * L_2 + \{\alpha, \beta, \gamma\} \cup \Xi_1 \cup \Xi_2 \cup \Xi_3,$$

where

$$\Xi_1 = \{\Diamond_2 \Diamond_2 p \land \Diamond_2 \varphi \to \Diamond_2(\Diamond_2 p \land \varphi) \mid \varphi \textit{ is a c-formula}\},$$
$$\Xi_2 = \{\Diamond_2 \Box_2 \bot \land \Diamond_2 \varphi \to \Diamond_2(\Box_2 \bot \land \varphi) \mid \varphi \textit{ is a c-formula}\},$$
$$\Xi_3 = \{\Diamond_2^i \varphi \to \Box_2^j(\Diamond_2 \top \to \Diamond_2 \varphi) \mid i, j \geq 0,\ \varphi \textit{ is a c-formula}\},$$

and by a c-formula *we mean a closed modal formula without $\Box_2$.*

Theorem 3. *If L_1 and L_2 are Horn axiomatizable Kripke complete logics, then $L_1 \rhd L_2 = L_1 \rhd_s L_2$.*

References

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