TiRS graphs and frames: a new setting for duals of canonical extensions of lattices

A.P.K. Craig[1], M.J. Gouveia[2], and M. Haviar*[3]

[1] University of Johannesburg, Auckland Park, 2006, South Africa,
[2] University of Lisbon & CAUL, P-1749-016 Lisbon, Portugal,
[3] M. Bel University, 974 01 Banská Bystrica, Slovakia,
miroslav.haviar@umb.sk,
WWW home page: http://www.fpv.umb.sk/∼mhaviar/

Abstract. We consider properties of the graphs that arise as duals of bounded lattices in Ploščica's representation via maximal partial maps into the two-element set. We introduce TiRS graphs which abstract those duals of bounded lattices. We demonstrate their one-to-one correspondence with so–called TiRS frames which are a subclass of the class of RS frames introduced by Gehrke to represent perfect lattices. This yields a dual representation of finite lattices via finite TiRS frames, or equivalently finite TiRS graphs, which generalises the well-known Birkhoff dual representation of finite distributive lattices via finite posets. By using both Ploščica's and Gehrke's representations in tandem we present a new construction of the canonical extension of a bounded lattice. We present two open problems that can be of interest to researchers working in this area.

Keywords: bounded lattice, canonical extension, perfect lattice, TiRS graph, TiRS frame, RS frame

The canonical extension $\mathbf{L}^\delta$ of a bounded lattice $\mathbf{L}$ was first introduced by Gehrke and Harding [6] as the complete lattice of Galois-closed sets associated with a polarity between the filter lattice $\mathrm{Filt}(\mathbf{L})$ and the ideal lattice $\mathrm{Idl}(\mathbf{L})$ of $\mathbf{L}$. We refer to Gehrke and Vosmaer [7] for a survey of the theory of canonical extensions for lattice-based algebras, including a discussion of their important role in the semantic modelling of logics. For all concepts and results needed in our work we refer to Section 2 of [3] which can be used as our preliminaries and which is available online.

The variety $\mathcal{L}$ of bounded lattices is not finitely generated, thus no natural duality theory in the terms of Clark and Davey [1] is available for $\mathcal{L}$. There is a well-known representation for $\mathcal{L}$ due to Urquhart [9] which was later presented in the spirit of natural duality theory by Ploščica in [8]. In [2] we used Ploščica's topological representation for constructing the canonical extension of a bounded lattice $\mathbf{L} \in \mathcal{L}$. Roughly speaking, this was done by replacing in Priestley's representation for distributive $\mathbf{L}$, total maps from $\mathbf{L}$ into $\{0, 1\}$ by appropriate maximally-defined partial maps into $\{0, 1\}$, viewed either as a lattice or a partially ordered set. More precisely, Ploščica's dual of a lattice $\mathbf{L}$ is a graph with topology, $\mathrm{D}(\mathbf{L}) = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E, \mathcal{T})$, where $\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}})$ is the set of maximal partial homomorphisms from $\mathbf{L}$ into $\underline{\mathbf{2}}$, the graph relation E is given by $(f, g) \in E$ if and only if $f^{-1}(1) \cap g^{-1}(0) = \emptyset$ and the topology $\mathcal{T}$ has as a subbasis of closed sets the set $\{ V_a, W_a \mid a \in L \}$, with $V_a = \{ f \in \mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f(a) = 0 \}$ and $W_a = \{ f \in \mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}) \mid f(a) = 1 \}$. We use the notation $\mathrm{D}^\flat(\mathbf{L})$ to refer to the graph $(\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$. (We note that what we call a *graph* is usually referred to as a *digraph.*) The canonical extension of $\mathbf{L}$ constructed in [2] is then taken to be the lattice $\mathrm{C}(\mathrm{D}^\flat(\mathbf{L})) = \mathcal{G}^{\mathrm{mp}}(\mathrm{D}^\flat(\mathbf{L}), \overline{2})$ of all maximal partial E-preserving maps from $\mathrm{D}^\flat(\mathbf{L}) = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$ to the two-element graph $\overline{2} = (\{0, 1\}, \leq)$. For these special graphs $\mathbf{X} = \mathrm{D}^\flat(\mathbf{L})$, as well as for arbitrary graphs $\mathbf{X} = (X, E)$, the lattice order in the (complete) lattice $\mathrm{C}(\mathbf{X})$ of all maximal partial E-preserving maps from $\mathbf{X}$ into $\overline{2}$ is given by $\varphi \leq \psi$ iff $\varphi^{-1}(1) \subseteq \psi^{-1}(1)$ for all $\varphi, \psi \in \mathrm{C}(\mathbf{X})$ (see [2], [3]).

In our work presented in [4], which partly is a continuation of [2] and [3], we show that duals of bounded lattices can be viewed as so–called *TiRS graphs* as well as so–called *TiRS frames.* Firstly, TiRS-graphs are introduced as an abstraction of the graphs $\mathrm{D}^\flat(\mathbf{L})$ obtained from Ploščica's duals of bounded lattices $\mathbf{L}$. As the main result, a one-to-one correspondence between TiRS graphs and TiRS frames is shown. We note that TiRS frames are special RS frames; the latter were introduced by Gehrke [5] in her dual representation of so–called *perfect lattices.*

Then we prove that every finite RS frame is a TiRS frame and we point that this yields a dual representation between finite lattices and finite TiRS frames, or equivalently finite TiRS graphs, which generalises the well-known Birkhoff dual representation between finite distributive lattices and finite posets. We also generalise the descriptions of the completely join-irreducible and the completely meet-irreducible elements in the complete lattices $\mathrm{C}(\mathbf{X})$, which were presented for the graphs $\mathbf{X} = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), E)$ in [3], to arbitrary reflexive reduced graphs $\mathbf{X} = (X, E)$. Finally, we show that Ploščica's dual representation of bounded lattices and Gehrke's dual representation of perfect lattices can be used in tandem to provide a new construction of the canonical extension for an arbitrary bounded lattice.

We discuss several natural questions and present two open problems that can be of interest to researchers working in this area. The first natural question to ask is the following one: *Is every TiRS graph* $\mathbf{X} = (X, R)$ *of the form* $\mathrm{D}^\flat(\mathbf{L}) = (\mathcal{L}^{\mathrm{mp}}(\mathbf{L}, \underline{\mathbf{2}}), R)$ *for some bounded lattice* $\mathbf{L}$*?* Firstly note that every poset is a TiRS graph. A poset is said to be *representable* if it is the underlying poset of some Priestley space and hence the untopologized dual of some bounded distributive lattice. It is known that *non-representable* posets exist and hence *non-representable* TiRS graphs exist. Thus the answer to the first question is *no.* Now it is natural to pose the following problem:

Problem 1 Which TiRS graphs arise as duals of bounded lattices?

We proved that the RS frame associated to the canonical extension of a lattice is always a TiRS frame. Hence, by the Gehrke correspondence between RS frames and perfect lattices, one could ask whether also conversely: *Does* *every TiRS frame correspond to a perfect lattice that is the canonical extension* $\mathbf{L}^\delta$ *of some lattice* **L***?* Since a non-representable poset is a TiRS graph, its corresponding frame is a TiRS frame. However, it does not correspond to the canonical extension of any bounded (distributive) lattice **L**. Thus the answer is again *no.* A natural problem to ask now is the following one:

Problem 2 Consider the perfect lattice which corresponds to a TiRS frame. In addition to being perfect, what additional properties of the complete lattice arise as a result of it coming from an RS frame which is also a TiRS frame?

We proved that the classes of finite RS frames and finite TiRS frames are the same. From this we obtain the following result, which generalises the classic Birkhoff dual represenation of finite distributive lattices via finite posets.

Theorem 3 There exists a dual representation of arbitrary finite lattices via finite TiRS graphs.

We now wonder whether this representation of finite lattices via TiRS graphs could bring a new light to the famous problem which has been open for decades: *Is every finite lattice isomorphic to the congruence lattice of some finite algebra?*

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