Ramsey's Theorem for pairs in k colors in the hierarchy of classical principles

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Abstract. The purpose of this work is to study, from the viewpoint of first order arithmetic (we have no set variables, the only sets are the arithmetical sets), Ramsey's Theorem for pairs for a recursive assignment of k-many colors in order to find some principle of classical logic equivalent to it in HA. We proved that it is equivalent in HA to Σ^0_3-LLPO, that is the Limited Lesser Principle of Omniscience for Σ^0_3 formulas.

1 Comparing Ramsey's Theorem and sub-classical principles

Ramsey's Theorem for pairs in k-many colors [5] states that given any coloring over the edges of the complete graph with countably many nodes in k-many colors, there exists an infinite homogeneous set, i.e. there exists an infinite subset of the set H of nodes and a color number $i = 0, \ldots, k-1$ such that for any $x, y \in H$ the edge $\{x, y\}$ has color number i.

As shown in [1], between Classical Arithmetic and Heyting Arithmetic (from now on, HA) there is a hierarchy of classical principles. In [2] we provided a first order proof of Ramsey's Theorem for pairs in two colors in order to find the minimal classical principle which implies the "miniature" version of Ramsey we may express in HA. We proved that Ramse's Theorem for pairs with recursive assignments of two colors, that is, the case $k = 2$ in the statement above, is equivalent in HA to the sub-classical principle Σ^0_3-LLPO. This principle says: if a disjunction between Σ^0_3-unary predicates is always true, then some of them is always true. In order to compare Ramsey's Theorem with first order classical principles, we expressed it as a schema in the first order language of arithmetic, instead of using quantification over sets and functions as it is more usual: all sets we deal with are explicitly defined as arithmetical predicates. In particular, we formally define the homogeneous set whose existence is stated by the Ramsey's theorem as some unary Δ^0_3-arithmetical predicate.

We conjectured that Ramsey's Theorem for pairs in k-many colors for any $k \geq 2$ is still equivalent to Σ^0_3-LLPO in HA. The goal of this work is to prove this fact.

One possible application of our result could be to use classical realization [3] to give constructive proofs of some combinatorial corollaries of Ramsey's Theorem; another, a formalization of Ramsey's Theorem in HA, using a proof assistant.

2 A sketch of the equivalence proof for k colors

Since trivially $\mathrm{RT}^2_\mathrm{k} \implies \mathrm{RT}^2_2$ holds in HA, we have $\mathrm{RT}^2_\mathrm{k} \implies \Sigma^0_3$-LLPO. On the other hand in order to prove Σ^0_3-LLPO $\implies \mathrm{RT}^2_\mathrm{k}$ we need to apply non-trivial changes to the proof that we used in the two colors case.

In the case for two colors we modified Jockusch proof of Ramsey's Theorem, which appeared in [4]. Given a coloring $c : [\mathbb{N}]^2 \to k$ we say that a subset X of $\mathbb{N}$ defines a 1-coloring if for all $x \in X$, any two edges from x to some $y,\ z \in X$ have the same color. If X is infinite and defines a 1-coloring, thanks to the Pigeonhole Principle we define an infinite arithmetical subset Y of X whose points all have the same color. Y is the homogeneous set for c we are looking for. So we need to find an infinite set which defines a 1-coloring.

A tree T on $\mathbb{N}$ defines a 1-coloring with respect to T if for all $x \in T$ and for any two proper descendants $y,\ z$ of x in T, the edges from x to $y,\ z$ have the same color. Assume there exists some infinite binary tree T defining a 1-coloring with respect to T. The sub-classical principle Σ^0_3-LLPO implies König's Lemma in HA, and König's Lemma implies that T has an infinite branch B. B defines an infinite 1-coloring and so proves RT^2_k. Therefore a sufficient condition for RT^2_k is the existence of an infinite binary tree defining a 1-coloring. This kind of tree was introduced by Erdős and is called an Erdős tree.

Jockusch proof (that is itself a modification of Erdős Rado proof) and our proof differ in the definition of the Erdős tree T, even if the general idea is similar. In both Jockusch proof and our one, an infinite homogeneous set is obtained from an infinite set of nodes of the same color in an infinite branch of T. In Erdős-Rado and Jockusch proofs the Pigeonhole Principle is applied to an infinite branch obtained by König Lemma. To formalize this proof in HA we would have to use Σ^0_4-LLPO, which is more than Σ^0_3-LLPO. In our proof we define a special Erdős tree T which has only one infinite branch. This unique infinite branch has the property that if T has infinitely many edges with color c, then this branch has infinitely many edges with color c. So we may apply the Pigeonhole Principle on T, which is a Π^0_1-set. The Pigeonhole Principle for Π^0_1-sets follows from Σ^0_3-LLPO in HA. In this way we produce a proof of Ramsey for pairs and k-colors in HA which only requires the sub-classical principle Σ^0_3-LLPO.

References

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